18.112 Functions of a Complex Variable
Fall 2008

Solution for 18.112 ps 4

1(Prob 3 on P130).

Method 1. We only need to prove that these functions has no limit as z tends to infinity. We can prove this by constructing two sequence $\{z_n\}$ and $\{w_n\}$ of complex numbers such that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = \infty$$

but

$$\lim_{n\to\infty} f(z_n) \neq \lim_{n\to\infty} f(w_n).$$

• For

$$f(z) = e^z,$$

take

$$z_n = n,\ w_n = -n;$$

• For

$$f(z) = \sin z \quad \text{or} \quad f(z) = \cos z,$$

take

$$z_n = 2\pi n,\ w_n = 2\pi n + 1.$$

Method 2. By definition, we only need to prove that

$$\lim_{z\to 0} z^m f(\frac{1}{z}) \neq 0$$

for any (fixed) $m \in \mathbb{N}$. We can prove this by choosing one sequence $z_n \to 0$ such that

$$\lim_{n\to\infty} z_n^m f(\frac{1}{z_n}) \neq 0.$$

• For

$$f(z) = e^z,$$

take

$$z_n = 1/n.$$

• For

$$f(z) = \sin z \ \text{ or } f(z) = \cos z,$$

take

$$z_n = \frac{1}{ni}.$$

Method 3. In Midterm we proved that if $f(z)$ is analytic in $\mathbb{C}$ and has a nonessential singularity at ∞, then f is a polynomial. Now all the functions

$$f_1(z) = e^z, \ f_2(z) = \sin z, \ f_3(z) = \cos z$$

are analytic in $\mathbb{C}$, and none of them is polynomial (by Taylor expansion or by the number of zero points), so they have essential singularities at ∞.

2(Prob 4 on P133).

Solution: By the conditions we know that

$$f(z) = f(0) + zh(z),$$

where $h(z)$ is analytic in a neighborhood of 0, and

$$h(0) \neq 0.$$

Thus there is a small neighborhood $B_\varepsilon(0)$ such that h is analytic and nonzero in it. By Corollary 2 on Page 142, we can define a single-valued analytic function

$$\tilde{h}(z) = h(z)^{1/n}$$

on $B_\varepsilon(0)$. Let

$$g(z) = z\tilde{h}(z^n),$$

we get

$$\begin{aligned} f(z^n) &= f(0) + z^n h(z^n) \\ &= f(0) + g(z)^n \end{aligned}$$

in $B_\varepsilon(0)$.

Remark: We can drop the condition

$$f'(0) \neq 0.$$

Since 0 is always a zero point of $f(z) - f(0)$, we can either write

$$f(z) = f(0) + z^m h(z),$$

where $h(z)$ is analytic in a neighborhood of 0, and $h(0) \neq 0$; or have

$$f(z) \equiv f(0).$$

In the first case we can proceed as before with

$$g(z) = z^m \tilde{h}(z^n),$$

and the second case is trivial.

3(Prob 4 on P148).

Solution: Apply Corollary 2 in page 142 to analytic function

$$f(z) = z,$$

we see that single-valued analytic branch of $\log z$ can be defined in any simply connected region which does not contain the origin. Then we can define single-valued analytic branch of z^α and z^z by

$$z^\alpha = e^{\alpha \log z}$$

and

$$z^z = e^{z \log z}.$$