18.112 Functions of a Complex Variable
Fall 2008

Solution for 18.112 ps 3

1(Prob 2 on P83).
Solution: First we will prove the following two lemmas.

Lemma 1. Reflection carries circles to circles.
Proof: Reflection is the composition of two maps: conjugation

$$z \mapsto \bar{z}$$

and linear transformation

$$z \mapsto R^2/(z - \bar{a}) + a,$$

both maps will carry circles to circles, thus reflection carries circles to circles.

Lemma 2. Suppose circles C_1, C_2 are both symmetric with respect to a line l, and C_3 is the reflection image of C_1 with respect to C_2. Then C_3 is also symmetric with respect to the line l. (Compare this with the symmetry principle!)
Proof: Without loss of generality, we can suppose l is the $x-$axis, and the center of C_2 is the origin. Then $z \in C_1$ and $z^* \in C_3$ are related by

$$z^* \bar{z} = R^2.$$

So

$$\bar{z} \in C_1 \Longrightarrow \bar{z^*} \in C_3,$$

i.e. C_3 is symmetric with respect to l.

Now we return to our original problem. The symmetric point of z is

$$z^* = 1/(\bar{z} - 2) + 2.$$

• *Reflect the imaginary axis.* By lemma 1, the image is a circle. By lemma 2, the image circle is symmetric with respect to the $x-$axis. More over,

$$0^* = 3/2, \infty^* = 2,$$

both lie on the $x-$axis, so the image circle is

$$\left|z - \frac{7}{4}\right| = \frac{1}{4}.$$

• *Reflect the line* $x = y$. Now the line l is $x + y = 2$, and both

$$(1+i)^* = (3+i)/2 \;\text{ and }\; \infty^* = 2$$

lie on l. So the image circle is

$$\left|z - \frac{7+i}{4}\right| = \frac{\sqrt{2}}{4}.$$

• *Reflect the circle* $|z| = 1$. The line l is still the $x-$axis. By

$$1^* = 1, \;\; (-1)^* = 5/3,$$

we get the image circle

$$\left|z - \frac{4}{3}\right| = \frac{1}{3}.$$

2(Prob 3 on P88).

Solution: We consider the number of fixed points of S. If S has more than 2 fixed points, then it has to be identity map, which is automatically elliptic. Moreover, by equation (13) on page 86, the fixed point will always exist (may be the point ∞).

Now we suppose that S has two distinct fixed points a and b, then we have

$$\frac{S(z) - a}{S(z) - b} = k\frac{z-a}{z-b}.$$

Thus

$$\begin{aligned}
\frac{z-a}{z-b} = \frac{S^n(z) - a}{S^n(z) - b} \\
&= k\frac{S^{n-1}(z) - a}{S^{n-1}(z) - b} \\
&= k^2\frac{S^{n-2}(z) - a}{S^{n-2}(z) - b} \\
&= \cdots\cdots \\
&= k^n\frac{z-a}{z-b}.
\end{aligned}$$

So

$$k^n = 1,$$

which implies

$$|k| = 1$$

and S is elliptic.
(If one of a, b, say a, is infinity, then

$$S(z) - b = k(z - b).$$

By the same way above, we see that

$$|k| = 1$$

and S is elliptic.)

At last suppose S has only one fixed point a. Let

$$Tz = 1/(z - a) + a,$$

which maps a to ∞ and ∞ to a. Since T is one-to-one and onto, the map TST^{-1} has only one fixed point

$$Ta = \infty.$$

Thus

$$TST^{-1}z = cz + d$$

for some $c \neq 0$. I claim that $c = 1$ in this case, otherwise

$$f = d/(1 - c)$$

is another fixed point of TST^{-1}. Now

$$z = TS^nT^{-1}z = (TST^{-1})^n z = z + nd,$$

which implies

$$d = 0.$$

So

$$TST^{-1} = Id,$$

and

$$S = T^{-1}(Id)T = Id$$

is elliptic.

3(Prob 5 on P88).

Solution: First it is easy to check that two points $a, b \in \mathbb{C}$ corresponding to diametrically opposite points on the Riemann sphere if and only if

$$a\bar{b} = -1$$

(Check it!), thus

$$b = -1/\bar{a}.$$

Let T be a linear transformation which represent rotation $\tilde{T}$ of the Riemann sphere. Then $\tilde{T}$ has two fixed points, A, B, which are opposite points. Thus T has two fixed points, a and $-1/\bar{a}$. Now consider the C_2 circles,

$$\frac{|z-a|}{|z+1/\bar{z}|} = c.$$

Let Z be the point on the Riemann sphere corresponding to z, then

$$\begin{aligned}\frac{|Z-A|}{|Z-B|} &= \frac{2|z-a|/\sqrt{(1+|z|^2)(1+|a|^2)}}{2|z+1/\bar{a}|/\sqrt{(1+|z|^2)(1+|1/\bar{a}|^2)}} \\ &= \frac{|z-a|}{|z+1/\bar{z}|}\frac{\sqrt{1+|1/\bar{a}|^2}}{\sqrt{1+|a|^2}}\end{aligned}$$

is constant, i.e. the C_2 circles are mapped to the "latitudinal" circles on the Riemann sphere, which are invariant under the rotation $\tilde{T}$. So the C_2 circles are unchanged under the mapping T, which tells us that T is elliptic.

On the other hand, suppose T is elliptic and has two fixed points $a, -1/\bar{a}$, then T maps each C_2 circle into itself, and maps C_1 circle to some C_1', and the angle between C_1 and C_1' is $\arg k$(See paragraph 3 on page 86). So $\tilde{T}$ maps "latitudinal" circle to itself, and maps "longitudinal" circles to another "longitudinal" circle by rotating angle $\arg k$ since the stereographic projection is conformal. Thus $\tilde{T}$ acts on C_2 circle ("latitudinal" circle) by rotating a fixed angle $\arg k$, which implies that $\tilde{T}$ is a rotation which fixes A, B.

Thus all linear transformations which represent rotations of the Riemann sphere are exactly all elliptic linear transformations with fixed points a and $-1/\bar{a}$ (when $a = 0$, let $-1/\bar{a} = \infty$).

4(Prob 2 on P108).

Solution:

$$
\begin{aligned}
1). \int_{|z|=r} x dz &= \int_0^{2\pi} (r\cos t)(-r\sin t + ir\cos t)dt \\
&= \int_0^{2\pi} (-r^2\cos t\sin t + ir^2\cos^2 t)dt \\
&= \frac{r^2}{2}\int_0^{2\pi} [-\sin 2t + i(\cos 2t + 1)]dt \\
&= \frac{r^2}{2} \times i \times 2\pi \\
&= \pi r^2 i.
\end{aligned}
$$

$$
\begin{aligned}
2). \int_{|z|=r} x dz &= \int_{|z|=r} \frac{z + r^2/z}{2} dz \\
&= \int_{|z|=r} \frac{z}{2} dz + \int_{|z|=r} \frac{r^2}{2z} dz \\
&= \frac{r^2}{2}\int_{|z|=r} \frac{1}{z} dz \\
&= \frac{r^2}{2} \times 2\pi i \\
&= \pi r^2 i.
\end{aligned}
$$

5(Prob 4 on P108).

Solution:

$$
\begin{aligned}
\int_{|z|=1} |z-1||dz| &= \int_0^{2\pi} |e^{it} - 1||ie^{it}|dt \\
&= \int_0^{2\pi} |\cos t + i\sin t - 1|dt \\
&= \int_0^{2\pi} \sqrt{(\cos t - 1)^2 + \sin^2 t} \\
&= \int_0^{2\pi} \sqrt{2 - 2\cos t} dt \\
&= \int_0^{2\pi} \sqrt{4\sin^2(t/2)} dt \\
&= -4\cos(t/2)|_0^{2\pi} \\
&= 8.
\end{aligned}
$$