18.112 Functions of a Complex Variable
Fall 2008

Solution for 18.112 ps 1

1(Prob1 on P11).

Solution:

$$
\begin{aligned}
|a| < 1, |b| < 1 &\Longrightarrow (1 - a\bar{a})(1 - b\bar{b}) < 1 \\
&\Longrightarrow 1 - a\bar{a} - b\bar{b} + a\bar{a}b\bar{b} < 1 \\
&\Longrightarrow 1 + a\bar{a}b\bar{b} - a\bar{b} - \bar{a}b > a\bar{a} + b\bar{b} - a\bar{b} - \bar{a}b \\
&\Longrightarrow (1 - a\bar{b})(1 - \bar{a}b) > (a - b)(\bar{a} - \bar{b}) \\
&\Longrightarrow \left| \frac{a - b}{1 - \bar{a}b} \right| < 1.
\end{aligned}
$$

2(Prob4 on P11).

Solution:

• If there is a solution, then

$$
\begin{aligned}
2|c| &= |z - a| + |z + a| \\
&\geq |(z - a) - (z + a)| \\
&= 2|a|,
\end{aligned}
$$

i.e.

$$|c| \geq |a|.$$

On the other hand, if

$$|c| \geq |a|,$$

take

$$z_0 = \frac{|c|}{|a|} a,$$

then it is easy to check that z_0 is a solution. Thus the largest value of $|z|$ is $|c|$, with corresponding $z = z_0$.

• Use fundamental inequality and formula (8) on page 8, we can get

$$\begin{aligned} 4|c|^2 &= (|z+a| + |z-a|)^2 \\ &\le 2(|z+a|^2 + |z-a|^2) \\ &= 4(|z|^2 + |a|^2) \\ \Longrightarrow |z| &\ge \sqrt{|c|^2 - |a|^2}, \end{aligned}$$

which can be obtained with

$$z = i\frac{\sqrt{|c|^2 - |a|^2}}{|a|}a.$$

N.B. Geometrically,

$$|z-a| + |z+a| = 2|c|$$

represents a ellipse, with long axis $|c|$ and focus a. So the short axis is

$$\sqrt{|c|^2 - |a|^2},$$

and thus

$$\sqrt{|c|^2 - |a|^2} \le |z| \le |c|.$$

3(Prob 1 on P17).

Solution: Suppose

$$az + b\bar{z} + c = 0$$

is a line, then it has at least two different solutions, say, z_0, z_1. Thus,

$$\begin{aligned} & az_0 + b\bar{z}_0 + c = 0, \ az_1 + b\bar{z}_1 + c = 0 \\ \Longrightarrow & a(z_0 - z_1) = b(\bar{z}_1 - \bar{z}_0) \\ \Longrightarrow & |a| = |b|. \end{aligned}$$

Thus

$$a \ne 0$$

and there is a θ such that

$$b = ae^{i\theta}.$$

So

$$\begin{aligned} & az + b\bar{z} + c = 0 \\ \Longleftrightarrow & az + ae^{i\theta}\bar{z} + c = 0 \\ \Longleftrightarrow & z + e^{i\theta}\bar{z} + c/a = 0 \\ \Longleftrightarrow & e^{-i\frac{\theta}{2}}z + \overline{e^{-i\frac{\theta}{2}}z} + e^{-i\frac{\theta}{2}}c/a = 0. \end{aligned}$$

This equation has solution if and only if

$$e^{-i\frac{\theta}{2}}c/a \in \mathbb{R},$$

in which case the equation does represent a line, given by

$$2\mathrm{Re}(e^{-i\frac{\theta}{2}}z) = -e^{-i\frac{\theta}{2}}c/a.$$

Note that

$$\begin{aligned} & e^{-i\frac{\theta}{2}}c/a \in \mathbb{R} \\ \Longleftrightarrow & e^{-i\frac{\theta}{2}}c/a = \overline{e^{-i\frac{\theta}{2}}c/a} \\ \Longleftrightarrow & c/(ae^{i\theta}) = \overline{c/a} \\ \Longleftrightarrow & c/b = \overline{c/a}. \end{aligned}$$

So the condition in form of a, b, c is

$$|a| = |b| \quad \text{and} \quad c/b = \overline{c/a}.$$

4(Prob 5 on P17).(We need to suppose $|a| \neq 1$.)

Solution: Let P, Q be the points on the plane corresponding to a and $1/\bar{a}$. By

$$\frac{1}{\bar{a}} = \frac{a}{|a|^2}$$

we know that O, P, Q are on the same line. Suppose the circle intersect the unit circle at points R, S.(They Do intersect at two points!) Then

$$|\overrightarrow{OR}|^2 = 1 = |a||1/\bar{a}| = |\overrightarrow{OP}||\overrightarrow{OQ}|.$$

By elementary planar geometry, $\overrightarrow{OR}$ tangent to the circle through P, Q, i.e. the radii to the point of intersection are perpendicular. So the two circles intersect at right angle.