**18.103 Fall 2013**

# Orthonormal Bases

Consider an inner product space $V$ with inner product $\langle f, g \rangle$ and norm

$$\|f\|^2 = \langle f, f \rangle$$

**Proposition 1** *(Continuity) If $\|u_n - u\| \to 0$ and $\|v_n - v\| \to 0$ as $n \to \infty$, then*

$$\|u_n\| \to \|u\|; \quad \langle u_n, v_n \rangle \to \langle u, v \rangle.$$

*Proof.* Note first that since $\|v_n - v\| \to 0$,

$$\|v_n\| \le \|v_n - v\| + \|v\| \le M < \infty$$

for a constant $M$ independent of $n$. Therefore, as $n \to \infty$,

$$|\langle u_n, v_n \rangle - \langle u, v \rangle| = |\langle u_n - u, v_n \rangle + \langle u, v_n - v \rangle| \le M\|u_n - u\| + \|u\|\|v_n - v\| \to 0$$

In particular, if $u_n = v_n$, then $\|u_n\|^2 = \langle u_n, u_n \rangle \to \langle u, u \rangle = \|u\|^2$. $\square$

For $u$ and $v$ in $V$ we say that $u$ is perpendicular to $v$ and write $u \perp v$ if $\langle u, v \rangle = 0$. The *Pythogorean theorem* says that if $u \perp v$, then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 \tag{1}$$

**Definition 1** *$\varphi_n$ is called an* orthonormal *sequence, $n = 1, 2, \ldots$, if $\langle \varphi_n, \varphi_m \rangle = 0$ for $n \ne m$ and $\langle \varphi_n, \varphi_n \rangle = \|\varphi_n\|^2 = 1$.*

Suppose that $\varphi_n$ is an orthonormal sequence in an inner product space $V$. The following four consequences of the Pythagorean theorem (1) were proved in class (and are also in the text):

If $h = \sum_{n=1}^{N} a_n \varphi_n$, then

$$\|h\|^2 = \sum_{1}^{N} |a_n|^2. \tag{2}$$

If $f \in V$ and $s_N = \sum_{n=1}^{N} \langle f, \varphi_n \rangle \varphi_n$, then

$$\|f\|^2 = \|f - s_N\|^2 + \|s_N\|^2 \tag{3}$$

If $V_N = \text{span}\,\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$, then

$$\|f - s_N\| = \min_{g \in V_N} \|f - g\| \qquad \text{(best approximation property)} \tag{4}$$

If $c_n = \langle f, \varphi_n \rangle$, then

$$\|f\|^2 \geq \sum_{n=1}^{\infty} |c_n|^2 \qquad \text{(Bessel's inequality).} \tag{5}$$

**Definition 2** *A Hilbert space is defined as a complete inner product space (under the distance $d(u, v) = \|u - v\|$).*

**Theorem 1** *Suppose that $\varphi_n$ is an orthonormal sequence in a Hilbert space $H$. Let*

$$V_N = \mathit{span}\,\{\varphi_1, \varphi_2, \ldots, \varphi_N\}, \quad V = \bigcup_{N=1}^{\infty} V_N$$

*($V$ is the vector space of finite linear combinations of $\varphi_n$.) The following are equivalent.*

*a) $V$ is dense in $H$ (with respect to the distance $d(f, g) = \|f - g\|$),*

*b) If $f \in H$ and $\langle f, \varphi_n \rangle = 0$ for all $n$, then $f = 0$.*

*c) If $f \in H$ and $s_N = \sum_{n=1}^{N} \langle f, \varphi_n \rangle \varphi_n$, then $\|s_N - f\| \to 0$ as $N \to \infty$.*

*d) If $f \in H$, then*

$$\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, \varphi_n \rangle|^2$$

If the properties of the theorem hold, then $\{\varphi_n\}_{n=1}^{\infty}$ is called an *orthonormal basis* or *complete orthonormal system* for $H$. (Note that the word "complete" used here does not mean the same thing as completeness of a metric space.)

*Proof.* (a) $\implies$ (b). Let $f$ satisfy $\langle f, \varphi_n \rangle = 0$, then by taking finite linear combinations, $\langle f, v \rangle = 0$ for all $v \in V$. Choose a sequence $v_j \in V$ so that $\|v_j - f\| \to 0$ as $j \to \infty$. Then by Proposition 1 above

$$0 = \langle f, v_j \rangle \to \langle f, f \rangle \implies \|f\|^2 = 0 \implies f = 0$$

(b) $\implies$ (c). Let $f \in H$ and denote $c_n = \langle f, \varphi_n \rangle$, $s_N = \sum_1^N c_n \varphi_n$. By Bessel's inequality (5),

$$\sum_1^\infty |c_n|^2 \leq \|f\|^2 < \infty.$$

Hence, for $M < N$ (using (2))

$$\|s_N - s_M\|^2 = \left\| \sum_{M+1}^N c_n \varphi_n \right\|^2 = \sum_{M+1}^N |c_n|^2 \to 0 \quad \text{as} \quad M,\ N \to \infty.$$

In other words, $s_N$ is a Cauchy sequence in $H$. By completeness of $H$, there is $u \in H$ such that $\|s_N - u\| \to 0$ as $N \to \infty$. Moreover,

$$\langle f - s_N, \varphi_n \rangle = 0 \quad \text{for all } N \geq n.$$

Taking the limit as $N \to \infty$ with $n$ fixed yields

$$\langle f - u, \varphi_n \rangle = 0 \quad \text{for all } n.$$

Therefore by (b), $f - u = 0$.

(c) $\implies$ (d). Using (3) and (2),

$$\|f\|^2 = \|f - s_N\|^2 + \|s_N\|^2 = \|f - s_N\|^2 + \sum_1^N |c_n|^2, \qquad (c_n = \langle f, \varphi_n \rangle)$$

Take the limit as $N \to \infty$. By (c), $\|f - s_N\|^2 \to 0$. Therefore,

$$\|f\|^2 = \sum_1^\infty |c_n|^2$$

Finally, for (d) $\implies$ (a),

$$\|f\|^2 = \|f - s_N\|^2 + \sum_1^N |c_n|^2$$

Take the limit as $N \to \infty$, then by (d) the rightmost term tends to $\|f\|^2$ so that $\|f - s_N\|^2 \to 0$. Since $s_N \in V_N \subset V$, $V$ is dense in $H$. $\square$

**Proposition 2** *Let $\varphi_n$ be an orthonormal sequence in a Hilbert space $H$, and*

$$\sum |a_n|^2 < \infty, \quad \sum |b_n|^2 < \infty$$

*then*

$$u = \sum_{n=1}^{\infty} a_n \varphi_n, \quad v = \sum_{n=1}^{\infty} b_n \varphi_n$$

*are convergent series in $H$ norm and*

$$\langle u, v \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n} \tag{6}$$

*Proof.* Let

$$u_N = \sum_1^N a_n \varphi_n; \quad v_N = \sum_1^N b_n \varphi_n.$$

Then for $M < N$,

$$\|u_N - u_M\|^2 = \sum_M^N |a_n|^2 \to 0 \quad \text{as } M \to \infty$$

so that $u_N$ is a Cauchy sequence converging to some $u \in H$. Similarly, $v_N \to v$ in $H$ norm. Finally,

$$\langle u_N, v_N \rangle = \sum_{j,k=1}^{N} \langle a_j \varphi_j, b_k \varphi_k \rangle = \sum_{j,k=1}^{N} a_j \overline{b_k} \langle \varphi_j, \varphi_k \rangle = \sum_{j=1}^{N} a_j \overline{b_j}$$

since $\langle \varphi_j, f_k \rangle = 0$ for $j \neq k$ and $\langle f_j, f_j \rangle = 1$. Taking the limit as $N \to \infty$ and using the continuity property (1), $\langle u_N, v_N \rangle \to \langle u, v \rangle$, gives (6). □

If $H$ is a Hilbert space and $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal basis, then every element can be written

$$f = \sum_{n=1}^{\infty} a_n \varphi_n \quad \text{(series converges in norm)}$$

The mapping

$$\{a_n\} \mapsto \sum_n a_n \varphi_n$$

is a linear isometry from $\ell^2(\mathbb{N})$ to $H$ that preserves the inner product. The inverse mapping is

$$f \mapsto \{a_n\} = \{\langle f, \varphi_n \rangle\}$$

It is also useful to know that as soon as a linear mapping between Hilbert spaces is an isometry (preserves norms of vectors) it must also preserve the inner product. Indeed, the inner product function (of two variables $u$ and $v$) can be written as a function of the norm function (of linear combinations of $u$ and $v$). This is known as polarization:

**Polarization Formula.**

$$\langle u, v\rangle = a_1\|u + iv\|^2 + a_2\|u + v\|^2 + a_3\|u\|^2 + a_4\|v\|^2 \tag{7}$$

with

$$a_1 = i/2, \quad a_2 = 1/2, \quad a_3 = -(1+i)/2, \quad a_4 = -(i+1)/2$$

*Proof.*

$$\begin{aligned}\|u + iv\|^2 &= \langle u + iv, u + iv\rangle \\ &= \|u\|^2 + \langle iv, u\rangle + \langle u, iv\rangle + \|v\|^2 \\ &= \|u\|^2 + i(\langle v, u\rangle - \langle u, v\rangle) + \|v\|^2\end{aligned}$$

Similarly,

$$\|u + v\|^2 = \|u\|^2 + (\langle v, u\rangle + \langle u, v\rangle) + \|v\|^2$$

Multiplying the first equation by $i$ and adding to the second, we find that

$$i\|u + iv\|^2 + \|u + v\|^2 = (i + 1)\|u\|^2 + 2\langle u, v\rangle + (i + 1)\|v\|^2$$

Solving for $\langle u, v\rangle$ yields (7). □