18.102 Introduction to Functional Analysis
Spring 2009

TEST 2 FOR 18.102: 9:35 – 10:55, 9 APRIL, 2009.
WITH SOLUTIONS

For full marks, complete and precise answers should be given to each question but you are not required to prove major results.

1. Problem 1

Let H be a separable (partly because that is mostly what I have been talking about) Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Say that a sequence u_n in H converges weakly if (u_n, v) is Cauchy in $\mathbb{C}$ for each $v \in H$.

(1) Explain why the sequence $\|u_n\|_H$ is bounded.

Solution: Each u_n defines a continuous linear functional on H by

$$T_n(v) = (v, u_n),\ \|T_n\| = \|u_n\|, T_n : H \longrightarrow \mathbb{C}. \tag{A.1}$$

For fixed v the sequence $T_n(v)$ is Cauchy, and hence bounded, in $\mathbb{C}$ so by the 'Uniform Boundedness Principle' the $\|T_n\|$ are bounded, hence $\|u_n\|$ is bounded in $\mathbb{R}$.

(2) Show that there exists an element $u \in H$ such that $(u_n, v) \to (u, v)$ for each $v \in H$.

Solution: Since (v, u_n) is Cauchy in $\mathbb{C}$ for each fixed $v \in H$ it is convergent. Set

$$Tv = \lim_{n\to\infty}(v, u_n) \text{ in } \mathbb{C}. \tag{A.2}$$

This is a linear map, since

$$T(c_1v_1 + c_2v_2) = \lim_{n\to\infty} c_1(v_1, u_n) + c_2(v_2, u) = c_1Tv_1 + c_2Tv_2 \tag{A.3}$$

and is bounded since $|Tv| \le C\|v\|$, $C = \sup_n \|u_n\|$. Thus, by Riesz' theorem there exists $u \in H$ such that $Tv = (v, u)$. Then, by definition of T,

$$(u_n, v) \to (u, v)\ \forall\ v \in H. \tag{A.4}$$

(3) If e_i, $i \in \mathbb{N}$, is an orthonormal sequence, give, with justification, an example of a sequence u_n which is *not* weakly convergent in H but is such that (u_n, e_j) converges for each j.

Solution: One such example is $u_n = ne_n$. Certainly $(u_n, e_i) = 0$ for all $i > n$, so converges to 0. However, $\|u_n\|$ is not bounded, so the sequence cannot be weakly convergent by the first part above.

(4) Show that if the e_i form an orthonormal basis, $\|u_n\|$ is bounded and (u_n, e_j) converges for each j then u_n converges weakly.

Solution: By the assumption that (u_n, e_j) converges for all j it follows that (u_n, v) converges as $n \to \infty$ for all v which is a finite linear combination of the e_i. For general $v \in H$ the convergence of the Fourier-Bessell series for v with respect to the orthonormal basis e_j

$$v = \sum_k (v, e_k)e_k \tag{A.5}$$

shows that there is a sequence $v_k \to v$ where each v_k is in the finite span of the e_j. Now, by Cauchy's inequality

$$|(u_n, v) - (u_m, v)| \le |(u_n v_k) - (u_m, v_k)| + |(u_n, v - v_k)| + |(u_m, v - v_k)|. \tag{A.6}$$

Given $\epsilon > 0$ the boundedness of $\|u_n\|$ means that the last two terms can be arranged to be each less than $\epsilon/4$ by choosing k sufficiently large. Having chosen k the first term is less than $\epsilon/4$ if $n, m > N$ by the fact that (u_n, v_k) converges as $n \to \infty$. Thus the sequence (u_n, v) is Cauchy in $\mathbb{C}$ and hence convergent.

2. Problem 2

Suppose that $f \in \mathcal{L}^1(0, 2\pi)$ is such that the constants

$$c_k = \int_{(0,2\pi)} f(x)e^{-ikx}, \ k \in \mathbb{Z},$$

satisfy

$$\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty.$$

Show that $f \in \mathcal{L}^2(0, 2\pi)$.

Solution. So, this was a good bit harder than I meant it to be – but still in principle solvable (even though no one quite got to the end).

First, (for half marks in fact!) we know that the c_k exists, since $f \in \mathcal{L}^1(0, 2\pi)$ and e^{-ikx} is continuous so $fe^{-ikx} \in \mathcal{L}^1(0, 2\pi)$ and then the condition $\sum_k |c_k|^2 < \infty$ implies that the Fourier series does converge in $L^2(0, 2\pi)$ so there is a function

$$g = \frac{1}{2\pi} \sum_{k\in\mathbb{C}} c_k e^{ikx}. \tag{A.1}$$

Now, what we want to show is that $f = g$ a.e. since then $f \in \mathcal{L}^2(0, 2\pi)$.

Set $h = f - g \in \mathcal{L}^1(0, 2\pi)$ since $\mathcal{L}^2(0, 2\pi) \subset \mathcal{L}^1(0, 2\pi)$. It follows from (A.1) that f and g have the same Fourier coefficients, and hence that

$$\int_{(0,2\pi)} h(x)e^{ikx} = 0 \ \forall \ k \in \mathbb{Z}. \tag{A.2}$$

So, we need to show that this implies that $h = 0$ a.e. Now, we can recall from class that we showed (in the proof of the completeness of the Fourier basis of L^2) that these exponentials are dense, in the supremum norm, in continuous functions which vanish near the ends of the interval. Thus, by continuity of the integral we know that

$$\int_{(0,2\pi)} hg = 0 \tag{A.3}$$

for all such continuous functions g. We also showed at some point that we can find such a sequence of continuous functions g_n to approximate the characteristic function of any interval χ_I. It is not true that $g_n \to \chi_I$ uniformly, but for any integrable function h, $hg_n \to h\chi_I$ in $\mathcal{L}^1$. So, the upshot of this is that we know a bit more than (A.3), namely we know that

$$\int_{(0,2\pi)} hg = 0 \ \forall \text{ step functions } g. \tag{A.4}$$

So, now the trick is to show that (A.4) implies that $h = 0$ almost everywhere. Well, this would follow if we know that $\int_{(0,2\pi)} |h| = 0$, so let's aim for that. Here is the trick. Since $g \in \mathcal{L}^1$ we know that there is a sequence (the partial sums of an absolutely convergent series) of step functions h_n such that $h_n \to g$ both in $L^1(0, 2\pi)$ and almost everywhere and also $|h_n| \to |h|$ in both these senses. Now, consider the functions

$$s_n(x) = \begin{cases} 0 & \text{if } h_n(x) = 0 \\ \frac{\overline{h_n(x)}}{|h_n(x)|} & \text{otherwise.} \end{cases} \tag{A.5}$$

Clearly s_n is a sequence of step functions, bounded (in absolute value by 1 in fact) and such that $s_n h_n = |h_n|$. Now, write out the wonderful identity

$$|h(x)| = |h(x)| - |h_n(x)| + s_n(x)(h_n(x) - h(x)) + s_n(x)h(x). \tag{A.6}$$

Integrate this identity and then apply the triangle inequality to conclude that

$$\begin{aligned} \int_{(0,2\pi)} |h| &= \int_{(0,2\pi)} (|h(x)| - |h_n(x)| + \int_{(0,2\pi)} s_n(x)(h_n - h) \\ &\leq \int_{(0,2\pi)} (||h(x)| - |h_n(x)|| + \int_{(0,2\pi)} |h_n - h| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{A.7}$$

Here on the first line we have used (A.4) to see that the third term on the right in (A.6) integrates to zero. Then the fact that $|s_n| \leq 1$ and the convergence properties.

Thus in fact $h = 0$ a.e. so indeed $f = g$ and $f \in \mathcal{L}^2(0, 2\pi)$. Piece of cake, right! Mia culpa.

3. Problem 3

Consider the two spaces of sequences

$$h_{\pm 2} = \{c : \mathbb{N} \longmapsto \mathbb{C}; \sum_{j=1}^{\infty} j^{\pm 4} |c_j|^2 < \infty\}.$$

Show that both $h_{\pm 2}$ are Hilbert spaces and that any linear functional satisfying

$$T : h_2 \longrightarrow \mathbb{C},\ |Tc| \leq C \|c\|_{h_2}$$

for some constant C is of the form

$$Tc = \sum_{j=1}^{\infty} c_i d_i$$

where $d : \mathbb{N} \longrightarrow \mathbb{C}$ is an element of h_{-2}.

Solution: Many of you hammered this out by parallel with l^2. This is fine, but to prove that $h_{\pm 2}$ are Hilbert spaces we can actually use l^2 itself. Thus, consider the maps on complex sequences

$$(T^{\pm} c)_j = c_j j^{\pm 2}. \tag{A.1}$$

Without knowing anything about $h_{\pm 2}$ this is a bijection between the sequences in $h_{\pm 2}$ and those in l^2 which takes the norm

$$\|c\|_{h_{\pm 2}} = \|Tc\|_{l^2}. \tag{A.2}$$

It is also a linear map, so it follows that $h_\pm$ are linear, and that they are indeed Hilbert spaces with $T^\pm$ isometric isomorphisms onto l^2; The inner products on $h_{\pm 2}$ are then

$$(c,d)_{h_{\pm 2}} = \sum_{j=1}^{\infty} j^{\pm 4} c_j \overline{d_j}. \tag{A.3}$$

Don't feel bad if you wrote it all out, it is good for you!

Now, once we know that h_2 is a Hilbert space we can apply Riesz' theorem to see that any continuous linear functional $T : h_2 \longrightarrow \mathbb{C}$, $|Tc| \le C\|c\|_{h_2}$ is of the form

$$Tc = (c, d')_{h_2} = \sum_{j=1}^{\infty} j^4 c_j \overline{d'_j},\ d' \in h_2. \tag{A.4}$$

Now, if $d' \in h_2$ then $d_j = j^4 d'_j$ defines a sequence in h_{-2}. Namely,

$$\sum_j j^{-4}|d_j|^2 = \sum_j j^4 |d'_j|^2 < \infty. \tag{A.5}$$

Inserting this in (A.4) we find that

$$Tc = \sum_{j=1}^{\infty} c_j d_j,\ d \in h_{-2}. \tag{A.6}$$