

## Practice Midterm 2 Solutions, 18.100C, Fall 2012

### 1

The statement is true. Let  $b_k = 2^{-k}a_k$ . By Rudin Theorem 3.23,  $\lim_{k \rightarrow \infty} a_k = 0$ . In particular, the  $a_k$ 's are bounded, i.e. for some  $a > 0$ ,  $|a_k| < a$ . Then for any  $k$ ,  $|b_k| < 2^{-k}a$ . But  $\sum_{k=0}^{\infty} 2^{-k}a = 2a$ , so  $|b_k|$  is bounded above by a convergent series, hence  $\sum_k |b_k|$  converges, and hence so does  $\sum_k b_k$ .

### 2

If  $f$  is constant  $f(x) := a$ , it is obviously periodic and its image is the one-point closed interval  $[a, a]$ . So assume  $f$  is non-constant.

Note that by induction,  $f(x + C) = f(x)$  implies  $f(x + nC) = f(x)$  for all positive integers  $n$ . We also have  $f(x) = f((x - C) + C) = f(x - C)$ , and so by induction  $f(x - nC) = f(x)$  for all positive integers  $n$ . Putting these together,  $f(x + nC) = f(x)$  for all  $n \in \mathbb{Z}$ .

Now let  $a \in \mathbb{R}$ . I claim there exists  $N \in \mathbb{Z}$  such that  $a - NC \in [0, C]$ . Indeed, let  $E = \{n \in \mathbb{Z} \mid nC < a\}$ . By the Archimedean property of the reals,  $E$  is bounded from above; by applying the Archimedean property to  $-a$  if  $a$  is negative, we see that  $E$  is also non-empty (since  $nC > -a \implies -nC < a$ ). Let  $N = \sup E$ .  $E$  is a closed (indeed, discrete) subset of the reals, so  $N \in E$ , and so  $NC < a$ , or  $a - NC > 0$ . On the other hand,  $a - NC \leq C$ , since otherwise  $a > (N + 1)C$ , contradicting  $N = \sup E$ .

Using this, I claim that  $f(\mathbb{R}) = f([0, C])$ . The inclusion  $f([0, C]) \subset f(\mathbb{R})$  is immediate. For the other inclusion, if  $b \in f(\mathbb{R})$ , then  $b = f(a)$ . Pick  $N$  as in the previous paragraph, and then  $a - NC \in [0, C]$  and  $f(a - NC) = f(a) = b$ .

The interval  $[0, C]$  is compact, so the image  $f([0, C])$  is also compact. Let

$c = \inf f([0, C])$  and  $d = \sup f([0, C])$ . Since compact sets are bounded,  $-\infty < c \leq d < \infty$ , and since they are closed  $c \in f([0, C])$  and  $d \in f([0, C])$ .

I claim that  $f([0, C]) = [c, d]$ . In one direction, suppose  $y \in f([0, C])$ . By the definition of inf and sup we have  $c \leq y \leq d$ , which is to say that  $y \in [c, d]$ . Hence  $f([0, C]) \subset [c, d]$ .

For the other direction, note that in fact  $c < d$ . Indeed, if  $c = d$ , then  $f(\mathbb{R}) = f([0, C]) \subset [c, c] = \{c\}$ , which means that  $f$  is constant, contra our assumptions. Now pick any  $a \in [0, C]$  with  $f(a) = c$ , and  $b \in [0, C]$  with  $f(b) = d$ ; we have  $a \neq b$ . Without loss of generality assume  $a < b$ ; the exact same argument will work for  $a > b$ .

Let  $y \in [c, d]$ .  $f(a) = c$  and  $f(b) = d$ , so apply the Intermediate Value Theorem to  $f$  restricted to  $[a, b]$ ; we conclude there exists  $x \in [a, b]$  with  $f(x) = y$ . Hence  $[c, d] \subset f([0, C])$  so  $f(\mathbb{R}) = [c, d]$ .

If you examine the above argument carefully, it actually proves that the continuous image of any closed and bounded interval is another closed and bounded interval. The only place where we used periodicity of  $f$  was to show that the image of  $f$  was the same as the image of some closed subinterval.

### 3

Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is uniformly continuous. Then there exists  $\delta > 0$  such that  $|x - y| < 2\delta \implies |f(x) - f(y)| < 1$ . For  $n \in \mathbb{Z}$ , let  $a_n = n\delta$ . Then we have

Claim:  $|f(a_n)| < |f(0)| + |n|$ .

Proof: we proceed by induction.  $a_0 = 0$ , so the base case is clear. Suppose we have proved this estimate for  $n \geq 0$ . Then  $|a_{n+1} - a_n| = \delta$ , and so  $|f(a_{n+1}) - f(a_n)| < 1$ . Then using the triangle inequality

$$|f(a_{n+1})| \leq |f(a_n)| + |f(a_{n+1}) - f(a_n)| < |f(0)| + n + 1$$

The argument for negative  $n$  is similar.

Now let  $C = |f(0)| + 1$  and  $D = 1/\delta$ . I claim that  $|f(x)| < C + D|x|$

for all  $x \in \mathbb{R}$ . First note that if  $x = a_n$  for some  $n$ , then  $|x| = |n|\delta$ , so  $D|x| = |n|$ , so in this case  $|f(x)| < C + D|x|$  follows from the claim.

Now let  $x \in \mathbb{R}$  be any point. Then take  $n \in \mathbb{Z}$  such that  $|x| \geq |a_n|$ , but  $|x - a_n| < \delta$ . The proof that such an  $n$  exists is very similar to the argument in the previous problem; if  $x > 0$ , one takes  $n$  to be the sup of all  $k$  such that  $a_k \leq x$ , while if  $x < 0$  one takes the inf of all  $k$  such that  $a_k \geq x$ . Then the first condition is satisfied, and if the second condition were not one could take  $a_{n+1}$  in the positive case or  $a_{n-1}$  in the negative one and produce a contradiction.

Thus we have  $|x - a_n| < \delta$ , so  $|f(x) - f(a_n)| < 1$ , and so

$$|f(x)| \leq |f(a_n)| + |f(x) - f(a_n)| < f(0) + n + 1 = f(0) + 1 + D|a_n| < C + D|x|$$

## 4

Note that  $f$  being twice differentiable at  $p$  means that  $f$  must be differentiable on some interval containing  $p$ . Since  $f'(p) = 0$ , we have

$$\lim_{x \rightarrow p} \frac{f'(x)}{x - p} = \frac{f'(x) - f'(p)}{x - p} = f''(p) > 0$$

Take  $\delta > 0$  such that  $f'(x)$  exists on  $[p - \delta, p + \delta]$  and

$$|x - p| < \delta, x \neq p \implies \left| f''(p) - \frac{f'(x)}{x - p} \right| < f''(p)/2$$

Then for  $|x - p| < \delta$ ,  $x \neq p$  we have  $f'(x)/(x - p) > f''(p)/2 > 0$ . If  $x > p$ , the denominator is positive, we must have  $f'(x) > 0$ , while if  $x < p$  then the denominator is negative and  $f'(x) < 0$ .

I claim that for  $|x - p| < \delta$ ,  $f(x) > f(p)$ . There are two cases to consider, although the arguments are of course almost identical.

$x > p$ :  $f$  is differentiable on the interval  $[p, x]$  by construction. Hence by the Mean Value Theorem there exists  $y \in (p, x)$  such that  $f(x) - f(p) = f'(y)(x - p)$ . Then  $|y - p| < |x - p| < \delta$  and  $y > p$ , and so  $f'(y) > 0$ , and  $f(x) - f(p) > 0$ .

$x < p$ :  $f$  is differentiable on the interval  $[x, p]$  by construction. Hence by the Mean Value Theorem there exists  $y \in (x, p)$  such that  $f(p) - f(x) = f'(y)(p - x)$ . Then  $|y - p| < |x - p| < \delta$  and  $y < p$ , and so  $f'(y) < 0$ , and  $f(p) - f(x) < 0$

In either case  $f(p) < f(x)$  and we are done.

Note that if  $f''$  exists and is continuous on a neighbourhood of  $p$ , there is a shorter, and perhaps more intuitive, proof. In this case, pick  $\delta > 0$  such that  $f'' > 0$  on  $[p - \delta, p + \delta]$ , which is possible since  $f''$  is continuous. Then for  $|x - p| < \delta$ , we have by the second order Taylor Formula

$$f(x) = f(p) + \frac{f''(y)}{2}(x - p)^2$$

For some  $y$  between  $x$  and  $p$ . But for such  $y$ ,  $f''(y) > 0$ , and we see that  $f(x) > f(p)$ . However, we were given only that  $f''(p)$  exists, so the longer proof was required.

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