Practice final exam solutions

1

We write $d = d_{SNCF}$ for the French Railroad metric on $\mathbb{R}^2$. In this problem we will often use the easily checked fact that if (x_n) is a Cauchy sequence in any metric space, then $x_n \to x$ if and only if $x_{n_k} \to x$ for some subsequence (x_{n_k}).

So let $(x_n) \subset \mathbb{R}^2$ be a Cauchy sequence with respect to d. We will show that (x_n) is convergent, and hence that d is a complete metric. If $x_n \to 0$ then obviously we are done, so assume (x_n) does not converge to 0. Note that since 0 lies on every line through the origin by definition, we have that $d(x_n, 0) = |x_n|$.

I claim that there exists $\epsilon > 0$ such that $|x_n| > \epsilon$ for all $n \in \mathbb{N}$. Indeed, if not then there exists a subsequence (x_{n_k}) with $|x_{n_k}| \to 0$ as $k \to \infty$, which means that the subsequence (x_{n_k}) converges to 0 with respect to d. Then since (x_n) is Cauchy, $x_n \to 0$, contradiction.

Now let $N \in \mathbb{N}$ be sufficiently large that $d(x_n, x_m) < \epsilon$ for $n, m > N$. Then x_n and x_m must lie on the same line. If they didn't, then

$$\epsilon > d(x_n, x_m) = |x_n| + |x_m| > \epsilon + \epsilon$$

Contradiction.

In other words, there exists a line $l \subset \mathbb{R}^2$ with $x_n \in l$ for $n > N$. For any $n, m > N$, we then have $d(x_n, x_m) = |x_n - x_m|$. Thus, (x_n) is a Cauchy sequence with respect to the standard Euclidean metric on $\mathbb{R}^2$, since d will agree that metric for sufficiently large n. Since $|\cdot|$ is complete, there exists

$x \in \mathbb{R}^2$ with $\lim_{n\to\infty} |x - x_n| = 0$. Lines are closed subsets of $\mathbb{R}^2$ with respect to $|\cdot|$, so $x \in l$. Then $d(x_n, x) = |x_n - x|$. Thus $x_n \to x$ with respect to d, and so d is complete.

2

Let $E = \mathbb{Q} \cap [0,1]$. E is an infinite subset of a countably infinite set, hence is countably infinite. In other words, there exists a bijective function $f : \mathbb{N} \to E$. Define the sequence (x_n) via $x_n = f(n)$. Note that $\overline{E} = [0,1]$

Let F be the set of all subsequential limits of E. I claim that $F = [0,1]$. Suppose that $x \in F$. Take a subsequence (x_{n_k}) converging to x. Then every neighbourhood of x contains all but finitely many (x_{n_k}), and in particular intersects E. So $x \in \overline{E} = [0,1]$.

Conversely, suppose $x \in [0,1]$. We will construct a subsequence $x_{n_k} \to x$ inductively. Let $n_1 = 1$. Suppose we have definted $n_1, n_2, \ldots n_k$. Consider the subset $A_{k+1} \subset \mathbb{N}$, defined by

$$A_{k+1} = \{n \in \mathbb{N} | |x - f(n)| < \frac{1}{k}\}$$

Recall that $x_n = f(n)$. Since x is a limit point of E, $B_{1/k+1}(x)$ contains infinitely many points of E; since f is surjective, this implies that A_{k+1} is infinite. Thus we can pick a $n_{k+1} \in A_{k+1}$ with $n_{k+1} > n_k$.

Thus we have constructed a subsequence (x_{n_k}) with $|x - x_{n_k}| < 1/k$, which means that $x_{n_k} \to x$, so $x \in F$. Thus $F = [0,1]$ and we are done.

3

We have a continuous function $f : [0,1] \times [0,1] \to \mathbb{R}$. For a fixed $x \in [0,1]$, consider the function $h_x : [0,1] \to \mathbb{R}$ defined by $h_x(y) = f(x,y)$. Then h_x is continuous; indeed, for any $y \in [0,1]$ and $\epsilon > 0$, take a $\delta > 0$ that works for f and ϵ at (x,y). Since h_x is a continuous function on a compact set, it attains a finite maximum; in other words, for some $y_0 \in [0,1]$, we have $f(x, y_0) = h_x(y_0) \geq h_x(y) = f(x,y)$ for all $y \in [0,1]$. Then

$$g(x) = \sup_{y\in[0,1]} \{f(x,y)\} = f(x, y_0)$$

Is well defined. We need to show that $g : [0,1] \to \mathbb{R}$ is continuous. Note that we have not only proved that g is well defined, but have also shown that for any $x \in [0,1]$, there exists $y \in [0,1]$ with $g(x) = f(x,y)$.

Let $x \in [0,1]$. We need to show that $\lim_{z \to x} g(z) = g(x)$. So suppose this is false. Then there exists $\epsilon > 0$ and a sequence (x_n) with $x_n \to x$ but $|g(x_n) - g(x)| > \epsilon$.

For each x_n, pick $y_n \in [0,1]$ such that $g(x_n) = f(x_n, y_n)$. Now consider the sequence $((x_n, y_n))_{n \in \mathbb{N}}$. This a sequence in the compact set $[0,1] \times [0,1]$, hence has a convergent subsequence. In other words there exists $(x', y') \in [0,1] \times [0,1]$ with $(x_{n_k}, y_{n_k}) \to (x', y')$. This implies that $x_{n_k} \to x'$, but since this a subsequence of a convergent sequence, it must also converge to x, and so $x' = x$.

f is continuous, and so

$$f(x, y') = \lim_{k \to \infty} f(x_{n_k}, y_{n_k}) = \lim_{k \to \infty} g(x_{n_k})$$

Hence, we must have $|g(x) - f(x,y')| \geq \epsilon$. Pick $y \in [0,1]$ with $f(x,y) = g(x)$. Then $f(x,y) \geq f(x,y')$, by the definition of g, and so

$$f(x,y) - f(x,y') \geq \epsilon$$

On the other hand, f is uniformally continuous, since it is a continuous function on a compact set. Pick a $\delta > 0$ such that

$$d((z,w),(z',w')) < \delta \implies d(f(z,w), f(z',w')) < \epsilon/3$$

and k sufficiently large that $|x - x_{n_k}|, |y' - y_{n_k}| < \delta/\sqrt{2}$. Then $|f(x,y) - f(x_{n_k}, y)| < \epsilon/3$, and so

$$f(x_{n_k}, y) > f(x,y) - \epsilon/3$$

Similarly, $|f(x_{n_k}, y_{n_k}) - f(x,y')| < \epsilon/3$, and so

$$f(x,y') + \epsilon/3 > f(x_{n_k}, y_{n_k})$$

Putting these together, we have

$$f(x_{n_k}, y) > f(x, y) - \epsilon/3 > f(x, y') + \epsilon/3 > f(x_{n_k}, y_{n_k})$$

This is a contradiction, since $f(x_{n_k}, y_{n_k}) = g(x_{n_k}) = \sup_y \{f(x_{n_k}, y)\}$.

4

We will show that $g'(x_0) = f''(x_0)/2$ by directly evaluating the limit of difference quotients. We have

$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0) - (x - x_0)f'(x)}{(x - x_0)^2}$$

Note that both the numerator and the denominator of the above expression converge to 0 as $x \to x_0$. Since f is twice differentiable at x_0, it must be once differentiable in some neighbourhood of x_0, otherwise the second derivative would not even make sense. Thus we can apply L'Hopital's rule; the derivative of the numerator is $f'(x) - f'(x_0)$, while the derivative of the denominator is $2(x - x_0)$. In other words, we have

$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{2(x - x_0)} = \frac{f''(0)}{2}$$

In particular, the limit exists, i.e. g is differentiable at x_0.

5

f is integrable, with integral 0. Note that any closed interval $[x, y]$ with $x < y$ contains a point z with $f(z) = 0$. Since f is non-negative, this implies that for any partition P, we have $L(f, P) = 0$.

Let $\epsilon > 0$. We will find a partition P with the upper Riemann sum $U(f, P) < 2\epsilon$, which will prove the result. Consider the function $g : [\epsilon, 1] \to \mathbb{R}$, which is equal to$f$ restricted to the interval $[\epsilon, 1]$. g has only finitely many points of discontinuity, namely, the finitely many points of the form $1/n > \epsilon$ for

$n \in \mathbb{N}$. Hence by Rudin Theorem 6.10, g is integrable. Since all lower Riemann Sums of g are zero, we must have

$$\int_{\epsilon}^{1} g(x) = 0$$

In particular, there exists a partition P of $[\epsilon, 1]$ with $U(g, P) < \epsilon$.

Now consider the partition of $[0, 1]$ defined by $P' = P \cup \{0\}$. Then all but the first term of $U(f, P')$ is contained in $U(g, P)$. More precisely, we have

$$U(f, P') = (\sup_{x \in [0,\epsilon]} f(x))(\epsilon - 0) + U(g, P) < \epsilon + \epsilon = 2\epsilon.$$

Which proves the result.

6

Since $f : [0, 1] \to \mathbb{R}$ is integrable, it is bounded, i.e. $|f(x)| < M$ for all $x \in [0, 1]$. We may assume $M > 1$. Let $\epsilon > 0$. Let $\delta < \epsilon/(2M)$.

Note that $0 < 1-\delta < 1$, and so by Rudin Theorem 3.20 $\lim_{n\to\infty}(1-\delta)^n = 0$. Let N be sufficiently large that $n > N \implies (1 - \delta)^n < \delta$. Then for any $0 \leq x \leq 1 - \delta$ and any $n > N$, we have $x^n < \delta$. Hence for $n > N$, we have

$$\left|\int_{0}^{1-\delta} f(x)x^n dx\right| \leq \int_{0}^{1-\delta} |f(x)|x^n dx < \int_{0}^{1-\delta} M\delta dx < M\delta < \frac{\epsilon}{2}$$

On the other hand $x^n \leq 1$ for $x \in [0, 1]$, and so

$$\left|\int_{1-\delta}^{1} f(x)x^n dx\right| < \int_{1-\delta}^{1} |f(x)|x^n dx < \int_{1-\delta}^{1} M dx = M\delta < \frac{\epsilon}{2}$$

Putting these together, we have

$$\left|\int_{0}^{1} f(x)x^n dx\right| \leq \left|\int_{0}^{1-\delta} f(x)x^n dx\right| + \left|\int_{1-\delta}^{1} f(x)x^n dx\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\lim_n \int_0^1 f(x)x^n dx = 0$.

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