18.100B Problem Set 9 Solutions

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1) First we need to show that

$$f_n(x) = \frac{1}{nx+1}$$

converges pointwise but not uniformly on $(0,1)$. If we fix some $x \in (0,1)$, we have that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{nx+1} = 0.$$

Thus the f_n converge pointwise. However, consider $\varepsilon = \frac{1}{4}$, and let $N \in \mathbb{N}$. Then

$$f_N(\frac{1}{N}) = \frac{1}{N(\frac{1}{N}+1)} = \frac{1}{2} \not< \varepsilon.$$

So the f_n do not converge to zero uniformly. Now we consider

$$g_n = \frac{x}{nx+1}.$$

Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be larger than $\frac{1}{\varepsilon}$. Then for any $x \in (0,1)$, $M \geq N$, we have

$$g_M(x) = \frac{x}{Mx+1} = \frac{1}{M+\frac{1}{x}} < \frac{1}{M} \leq \frac{1}{N} < \varepsilon.$$

Thus, g_n converges to 0 uniformly on (0,1).

2) The functions f_n are defined on $\mathbb{R}$ by

$$f_n = \frac{x}{1+nx^2}.$$

First we show $f_n \longrightarrow 0$. For any $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon^2}$. Then for $n > N$, if $|x| \leq \varepsilon$, $|x| < \varepsilon(1+nx^2)$ so $|f_n(x)| < \varepsilon$. If $|x| > \varepsilon$,

$$|n\varepsilon x| > n\varepsilon^2 > 1 \Longrightarrow |n\varepsilon x^2| > |x| \Longrightarrow \varepsilon > \left|\frac{x}{nx^2}\right| > \left|\frac{x}{1+nx^2}\right| = |f_n(x)|.$$

Thus f_n converges uniformly to $0 = f$.

Differentiating, we find

$$f_n'(x) = \frac{(1+nx^2)\cdot 1 - x\cdot(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}.$$

Thus,

$$|f_n'(x)| = \left|\frac{1-nx^2}{(1+nx^2)^2}\right| = \frac{|1-nx^2|}{(1+nx^2)^2} \leq \frac{1+nx^2}{(1+nx^2)^2} = \frac{1}{1+nx^2}.$$

So for all $x \neq 0$, $f_n'(x) \longrightarrow 0$, but $f_n(0) = 1$, $\forall n \in \mathbb{N}$. So if $g(x) = \lim f_n(x)$, $g(0) = 1 \neq 0 = f'(0)$. But $f'(x) = 0$, and thus exists, for all x.

For all $x \neq 0$, $f'(x) = 0 = g(x)$. We showed above that $f_n \longrightarrow f$ uniformly on all of $\mathbb{R}$. Finally, $f_n' \longrightarrow g$ uniformly away from zero, that is, on any interval that does not have zero as a limit point. This is true because the denominator of f_n' can be made arbitrarily large compared to the numerator, for $|x| > \varepsilon$.

3) So we have $f_n \longrightarrow f$ uniformly for f_n bounded. Then for $\varepsilon = 1$, $\exists N$ such that $\forall n > N$, $x \in E$, $|f_n(x) - f(x)| < 1$. Thus if f_n is bounded by B_n, $\forall x \in E$, $|f(x)| < 1 + B_N$ which implies that $\forall n > N$, $x \in E$, $f_n(x) < B_N + 2$. Thus (f_n) is uniformly bounded by $\max\{B_1, B_2, ..., B_{N-1}, B_N + 2\}$.

If the f_n are converging pointwise, f need not be bounded. For example, on $(0, 1)$, if

$$f_n(x) = \frac{1}{x + \frac{1}{n}},$$

then $f_n \longrightarrow \frac{1}{x}$ pointwise, and each f_n is bounded by n, but of course $\frac{1}{x}$ is unbounded on $(0, 1)$.

4) We have $f_n \longrightarrow f$ and $g_n \longrightarrow g$ uniformly.

a) Given $\varepsilon > 0$, $\exists M, N \in \mathbb{N}$ such that for any $m > M$, $n > N$, $|f_m - f| < \frac{\varepsilon}{2}$ and $|g_n - g| < \frac{\varepsilon}{2}$. So for $n, m > \max\{M, N\}$

$$|(f_n + g_n) - (f + g)| = |(f_n - f) + (g_n - g)| \leq |f_n - f| + |g_n - g| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $(f_n + g_n) \longrightarrow (f + g)$ uniformly.

b) If each f_n and g_n is bounded on E, by the previous problem they are uniformly bounded, say by A and B. Say without loss of generality $A \geq B$. Then for $\varepsilon > 0$, choose N such that if $n > N$, $|f_n - f| < \frac{\varepsilon}{2A}$ and $|g_n - g| < \frac{\varepsilon}{2A}$. Then

$$|f_n g_n - fg| = |f_n g_n - f g_n + f g_n - fg| \leq |f_n - f||g_n| + |f||g_n - g| < \frac{\varepsilon}{2A} A + \frac{\varepsilon}{2A} A = \varepsilon.$$

So $f_n g_n$ converges to fg uniformly.

5) Define

$$f(x) = x, \qquad g(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ q & \text{if, in lowest terms, } x = p/q \end{cases}$$

so that

$$f_n(x) = f(x)\left(1 + \frac{1}{n}\right), \quad \text{and} \quad g_n(x) = g(x) + \frac{1}{n}.$$

On any interval $[a, b]$, with $M = \max(|a|, |b|)$ we have

$$|f_n(x) - f(x)| = \frac{|x|}{n} \leq \frac{M}{n}, \qquad |g_n(x) - g(x)| = \frac{1}{n}$$

so that $f_n \rightarrow f$ and $g_n \rightarrow g$ uniformly.

On the other hand, with $m = \min(|a|, |b|)$ we have

$$\begin{aligned} |f_n(x) g_n(x) - f(x) g(x)| &= |(f_n(x) - f(x)) g_n(x) + f(x)(g_n(x) - g(x))| \\ &= \frac{f(x)}{n}\left(g(x) + \frac{n+1}{n}\right) \geq \frac{m}{n} g(x). \end{aligned}$$

Now notice that if $L \in \mathbb{N}$ is larger than $b - a$ then there is an integer k such that $\frac{k}{L} \in [a, b]$, choosing $L \in \mathbb{N}$ larger than $\frac{n}{m}$ and larger than $b - a$ we get

$$\|f_n g_n - fg\| \geq \frac{m}{n} g\left(\frac{k}{L}\right) = \frac{m}{n} L > 1$$

and hence $f_n g_n$ does not converge to fg uniformly.

6) We want to show $g \circ f_n \longrightarrow g \circ f$ uniformly, for g continuous on $[-M, M]$. Since g is continuous on a compact set, it is uniformly continuous. So given $\varepsilon > 0$, $\exists \delta > 0$ such that if $|x - y| < \delta$, $|g(x) - g(y)| < \varepsilon$. Now since $f_n \longrightarrow f$ uniformly, $\exists N \in \mathbb{N}$ such that for any $n > N$, $x \in E$, $|f_n(x) - f(x)| < \delta$. Thus $|g(f_n(x)) - g(f(x))| < \varepsilon$, so $|g \circ f_n(x) - g \circ f(x)| < \varepsilon$. Thus we have shown $(g \circ f_n) \longrightarrow g \circ f$ uniformly on E.

7) a) First we claim that, for every $x \in [0, 1]$,

$$0 \leq P_n(x) \leq P_{n+1}(x) \leq \sqrt{x}.$$

This is clearly true for $n = 0$, so assume inductively that it is true for $n = k$ and notice that

$$\begin{aligned}\sqrt{x} - P_{k+1}(x) &= \sqrt{x} - \left[P_k(x) + \frac{1}{2}\left(x - P_k(x)^2\right)\right] \\ &= \sqrt{x} - P_k(x) - \frac{1}{2}\left(\sqrt{x} - P_k(x)\right)\left(\sqrt{x} + P_k(x)\right) \\ &= \left(\sqrt{x} - P_k(x)\right)\left(1 - \frac{1}{2}\left(\sqrt{x} + P_k(x)\right)\right) \geq 0.\end{aligned}$$

It follows that $P_{k+1}(x) \leq \sqrt{x}$, and then

$$P_{k+1}(x) - P_k(x) = \frac{1}{2}\left(x - P_k(x)^2\right) \geq 0$$

shows that $P_k(x) \leq P_{k+1}(x)$, and proves the claim.

Notice that for every fixed x, the sequence $(P_n(x))$ is monotone increasing and bounded above (by $\sqrt{x}$), it follows that this sequence converges, to say $f(x)$. This function $f(x)$ is non-negative and satisfies

$$f(x) = \lim_{n \longrightarrow \infty} P_{n+1}(x) = \lim_{n \longrightarrow \infty}\left[P_n(x) + \frac{1}{2}(x - P_n(x)^2)\right] = f(x) + (x - f(x)^2)$$

which implies $f(x) = \sqrt{x}$, and hence the polynomials converge pointwise to $\sqrt{x}$ on $[0, 1]$. Since they are continuous and converge monotonically to a continuous function on $[0, 1]$, a compact set, they converge uniformly (by Dini's theorem).

b) Here we use a something similar to problem 6 along with the above work to show that $P_n(x^2) \longrightarrow |x|$ on $[-1, 1]$. The difference between this and problem 6 is we need $(f_n \circ g) \longrightarrow (f \circ g)$, but this is easier. Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ with $\forall x \in [0, 1]$, $n > N$, $|P_n(x) - \sqrt{x}| < \varepsilon$. So for all $x \in [-1, 1]$, $n > N$, $|P_n(x^2) - \sqrt{x^2}| < \varepsilon$. Since $|x| = \sqrt{x^2}$, we have show that the polynomials $P_n(x^2)$ converge uniformly to $|x|$ on $[-1, 1]$.

18.100B Analysis I
Fall 2010