SOLUTIONS TO PS6

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Solution/Proof of Problem 1. *Since* $\sum |a_n|$ *converges,* $\lim\limits_{n\to\infty} |a_n| = 0$. *So* $\exists N \in \mathbb{N}$ *such that for* $n \geqslant N$, $|a_n| < 1$. *Thus for* $n \geq N$ *we have* $|a_n^2| \leq |a_n|$ *and by the comparison theorem and the convergence of* $\sum |a_n|$ *we conclude that* $\sum |a_n^2|$ *converges.*

Solution/Proof of Problem 2. *Notice that*

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right),$$

so

$$\begin{aligned}\sum_{n=1}^{m} \frac{1}{n(n+1)(n+2)} &= \sum_{n=1}^{m} \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right) \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(m+1)(m+2)}\right).\end{aligned}$$

Then $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$.

Solution/Proof of Problem 3.

a) $\sum\limits_{k=1}^{n} a_k = \sum\limits_{k=1}^{n} (\sqrt{k+1} - \sqrt{k}) = \sum\limits_{k=1}^{n} \sqrt{k+1} - \sum\limits_{k=1}^{n} \sqrt{k} = \sum\limits_{k=2}^{n+1} \sqrt{k} - \sum\limits_{k=1}^{n} \sqrt{k} = \sqrt{n+1} - 1$, *so diverges.*

b) Notice that

$$\frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} < \frac{1}{2n\sqrt{n}}$$

and $\sum \frac{1}{2n\sqrt{n}}$ *converges, hence by the comparison theorem so does* $\sum \frac{\sqrt{n+1}-\sqrt{n}}{n}$

c) When $\alpha > 1$, *we have* $\sum\limits_{k=1}^{n} \frac{1}{\alpha^k}$ *converges. So* $\sum\limits_{k=1}^{n} \frac{1}{1+\alpha^n}$ *is an increasing bounded sequence, so it converges.*

When $\alpha \leqslant 1$, *we have* $\sum\limits_{k=1}^{n} \frac{1}{1+\alpha^k} \geqslant \sum\limits_{k=1}^{n} \frac{1}{2} = \frac{n(n+1)}{2}$, *so* $\sum\limits_{k=1}^{n} \frac{1}{1+\alpha^n}$ *diverges.*

Solution/Proof of Problem 4. *The Cauchy-Schwarz inequality tells us that*

$$\left|\sum_{k=1}^{n} \sqrt{a_k}\frac{1}{k}\right| \leq \left(\sum_{k=1}^{n} (\sqrt{a_k})^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \left(\frac{1}{k}\right)^2\right)^{\frac{1}{2}}$$

and since both $\sum a_k$ *and* $\sum \frac{1}{k^2}$ *converge, we know that* $\sum_{k=1}^{n} \frac{\sqrt{a_k}}{k}$ *is a bounded increasing sequence, hence it must converge.*

Solution/Proof of Problem 5. *Notice that* $\sum a_n$ *converges so* $\lim\limits_{n\to\infty} a_n = 0$ *and since* a_n *is decreasing this implies* $a_n \geq 0$.

Now since

$$\sum_{k=n+1}^{2n} a_k \geqslant na_{2n} \geqslant 0,$$

take limits on both side, we have $\lim_{n\to\infty} na_{2n} = 0$ *i.e.* $\lim_{2n\to\infty} 2na_{2n} = 0$. *Similarly, we can prove* $\lim_{2n+1\to\infty} (2n+1)a_{2n+1} = 0$. *So*

$$\lim_{n\to\infty} na_n = 0.$$

Solution/Proof of Problem 6.

1) *First note that, for any function f and any set $B \subseteq Y$, it is always true that $X = f^{-1}(B) \cup f^{-1}(B^c)$, and since these sets are disjoint, we always have $f^{-1}(B^c) = f^{-1}(B)^c$.*
   (a) $\iff$ (b)
   Suppose (a) is true and B is closed in Y, then B^c is open in Y and (a) implies $f^{-1}(B^c) = f^{-1}(B)^c$ open in X and hence $f^{-1}(B)$ is open in X. This proves (a) $\implies$ (b), and exchanging the words open and closed we get a proof that (b) $\implies$ (a).
2) *Note that for any function f and any sets $A \subseteq X$, $B \subseteq Y$ we always have*

$$A \subseteq f^{-1}(f(A)), \qquad f\left(f^{-1}(B)\right) \subseteq B.$$

   The first inclusion is an equality precisely when f is one-to-one, while the second is an equality precisely when f is onto. (So for instance if f is constant and X and Y have more than one point, both inclusions are strict in general.)
   (b) $\implies$ (c)
   Suppose (b) is true and let A be any subset of X. We know that

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}\left(\overline{f(A)}\right)$$

   and that $f^{-1}\left(\overline{f(A)}\right)$ is closed, hence $\overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right)$ and so

$$f\left(\overline{A}\right) \subseteq f\left(f^{-1}\left(\overline{f(A)}\right)\right) \subseteq \overline{f(A)}$$

   and (c) is true.
   (c) $\implies$ (b)
   Now suppose (c) is true, and $B \subseteq Y$ is closed. We can apply (c) to $A := f^{-1}(B)$ and get

$$f\left(\overline{A}\right) \subseteq \overline{f(A)} \iff f\left(\overline{f^{-1}(B)}\right) \subseteq \overline{f\left(f^{-1}(B)\right)}$$

   As mentioned above, $f\left(f^{-1}(B)\right) \subseteq B$ so we have

$$f\left(\overline{f^{-1}(B)}\right) \subseteq \overline{f\left(f^{-1}(B)\right)} \subseteq \overline{B} = B$$

   hence

$$\overline{f^{-1}(B)} \subseteq f^{-1}\left(f\left(\overline{f^{-1}(B)}\right)\right) \subseteq f^{-1}(B)$$

   which implies $f^{-1}(B)$ is closed, i.e. (b) is true.

Solution/Proof of Problem 7. *Consider the function*

$$f(x) = \begin{cases} 0 & \textit{if } x = 0 \\ 1 & \textit{if } x \neq 0 \end{cases}$$

Then it is easy to see that $f(x)$ is not continuous at the point $x = 0$. But it satisfies the condition in the problem. So that condition does **NOT** *imply f continuous.*

18.100B Analysis I
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