SOLUTIONS TO PS2

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Problem 1.

Proof. It is true that for any two sets A, B, the intersection $A \cap B$ is a subset of A. Now consider $\phi = A \cap A^c$. So ϕ is a subset of A for any set A. $\square$

Problem 2.

Proof. Notice that

$$||x| - |y|| \leq |x - y| \Leftrightarrow |x| - |y| \leq |x - y| \text{ and } |y| - |x| \leq |x - y|.$$

So we only need to prove that

$$|x| \leq |x - y| + |y| \text{ and } |y| \leq |x - y| + |x|.$$

But both of them is a consequence from the triangle inequality $|a - b| \leq |a - c| + |b - c|$. $\square$

Problem 3.

(a) $M = \{\frac{|x|}{1+|x|} : x \in \mathbb{R}\}$.

Proof. Notice that

$$\frac{|x|}{1 + |x|} = \frac{1}{\frac{1}{|x|} + 1}$$

so if $|x| < |y|$ then

$$\frac{|x|}{1 + |x|} < \frac{|y|}{1 + |y|}.$$

Thus the supremum is $\frac{1}{0+1} = 1$ and the infimum is $\frac{0}{1+0} = 0$. $\square$

(b) $M = \{\frac{x}{1+x} | x > -1\}$.

Proof. We can change the variable x to y,

$$\frac{x}{1 + x} = \frac{y - 1}{y} = 1 - \frac{1}{y},$$

where $y = x + 1$. From $x > -1$, we have $y > 0$. Notice that

$$y \text{ increases} \Rightarrow \frac{1}{y} \text{ decreases} \Rightarrow (1 - \frac{1}{y}) \text{ increases},$$

so the supremum is $1 - 0 = 1$ and the infimum is $-\infty$ (because for every $N > 1$ we have

$$\frac{\left(\frac{N}{1-N}\right)}{1 + \left(\frac{N}{1-N}\right)} = -N$$

and so the infimum is less than $-N$). $\square$

(c) $M = \{x + \frac{1}{x} | 1/2 < x < 2\}$.

Proof. It is always true that

$$\frac{a+b}{2} \geq \sqrt{ab},$$

for instance, if square both sides and rearrange, this is the same as saying $a^2+b^2 \geq 0$. Thus, we see that

$$x + \frac{1}{x} \geq 2\sqrt{x\frac{1}{x}} = 2$$

Since setting $x = 1$ in $x + \frac{1}{x}$ we get 2, we know that $\mathbf{inf} M = 2$.

Suppose we have $x_1 > x_2$, consider

$$\begin{aligned} & x_1 + \frac{1}{x_1} - (x_2 + \frac{1}{x_2}) \\ = \quad & (x_1 - x_2) + \frac{x_2 - x_1}{x_1 x_2} \\ = \quad & \frac{(x_1 - x_2)(x_1 x_2 - 1)}{x_1 x_2} \end{aligned}$$

So if $x_1, x_2 > 1$, then

$$x_1 + \frac{1}{x_1} - \left(x_2 + \frac{1}{x_2}\right) = \frac{(x_1 - x_2)(x_1 x_2 - 1)}{x_1 x_2} > 0,$$

i.e. $x + \frac{1}{x}$ is an increasing function; if $x_1, x_2 < 1$, then

$$x_1 + \frac{1}{x_1} - \left(x_2 + \frac{1}{x_2}\right) = \frac{(x_1 - x_2)(x_1 x_2 - 1)}{x_1 x_2} < 0,$$

i.e. $x+\frac{1}{x}$ is an decreasing function. Then the sup must be obtained at the boundary of $(1/2, 2)$.

Since

$$\lim_{x \to 2} \left(x + \frac{1}{x}\right) = \lim_{x \to 1/2} \left(x + \frac{1}{x}\right) = \frac{5}{2},$$

we have $\mathbf{sup} M = \frac{5}{2}$. □

Problem 4.

Proof. The answer is:

n	30	42	66	78	102	114	138	174	186	70	110
p1	2	2	2	2	2	2	2	2	2	2	2
p2	3	3	3	3	3	3	3	3	3	5	5
p3	5	7	11	13	17	19	23	29	31	7	11

n	130	170	190	154	182	105	165	195
p1	2	2	2	2	2	3	3	3
p2	5	5	5	7	7	5	5	5
p3	13	17	19	11	13	7	11	13

□

Problem 5.

Proof. From $X \sim \mathbb{R}$, then there is a $1-1$ mapping $\alpha : X \to \mathbb{R}$. Similarly we have a $1-1$ mapping $\beta : Y \to \mathbb{N}$. So to prove $Z = X \cup Y \sim \mathbb{R}$, we only need to prove that there is a $1-1$ mapping $\gamma : Z \to \mathbb{R}$. It is equivalent to show that there is $1-1$ mapping $\delta : \mathbb{N} \cup \mathbb{R} \to \mathbb{R}$. The δ can be constructed by the following method:

$$\begin{aligned} \delta(x) &= x && \text{, if } x \in \mathbb{R}\backslash\mathbb{Z}; \\ \delta(x) &= x && \text{, if } x \in \mathbb{Z} \subset \mathbb{R} \text{ and } \ x \leq 0; \\ \delta(x) &= 2x && \text{, if } x \in \mathbb{Z} \subset \mathbb{R} \text{ and } \ x > 0; \\ \delta(x) &= 2x+1 && \text{, if } x \in \mathbb{N}. \end{aligned}$$

It is easy to check that it is an $1-1$ mapping. □

Problem 6.

Proof. Consider the sets

$$\begin{aligned} A_0 &= \{\frac{1}{n}|n \in \mathbb{N}\}, \\ A_1 &= \{\frac{1}{n}+1|n \in \mathbb{N}\}, \\ A_2 &= \{\frac{1}{n}+2|n \in \mathbb{N}\}. \end{aligned}$$

Then A_i has only one limit point i, for $i = 0, 1, 2$. If let $A = \bigcup_{i=0}^{2} A_i$, we get a bounded set A with three limit points.

Consider the set

$$\mathcal{A} = \{\frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{N}\},$$

we check that the limit points of $\mathcal{A}$ are precisely the points in $A_0 \cup \{0\}$. Indeed, if we fix $n_0 \in \mathbb{N}$ then the set

$$\{\frac{1}{n_0} + \frac{1}{m} : m \in \mathbb{N}\},$$

has $\frac{1}{n_0}$ as a limit point and is a subset of $\mathcal{A}$, hence $\mathcal{A}$ has $\frac{1}{n_0}$ as a limit point, for any n_0 in $\mathbb{N}$. Also $A_0 \subseteq \mathcal{A}$ so zero is a limit point of $\mathcal{A}$. To see that there are no other limit points, pick a point $x \in \mathbb{R}$ that is not equal to $\frac{1}{n}$ for any $n \in \mathbb{N}$, we show that x is not a limit point of $\mathcal{A}$. We can find $N \in \mathbb{N}$ such that

$$\frac{1}{N} < x < \frac{1}{N-1}$$

Pick $\varepsilon > 0$ small enough so that

$$\frac{1}{N} < x - \varepsilon < x < x + \varepsilon < \frac{1}{N-1}$$

and notice that there are at most finitely many elements of $\mathcal{A}$ in $(x-\varepsilon, x+\varepsilon)$. Here is one way to see this: if n and m are both bigger than $2N$ then $\frac{1}{n} + \frac{1}{m} < \frac{1}{N}$, if $n < N$ then $\frac{1}{n} + \frac{1}{m} > \frac{1}{N-1}$, while if $2N \geq n > N$ then

$$\frac{1}{N} < \frac{1}{n} + \frac{1}{m} \iff -\frac{1}{m} < \frac{1}{n} - \frac{1}{N} = \frac{N-n}{nN} \iff m < \frac{nN}{n-N},$$

finally if $n = N$, and m is large enough then $\frac{1}{n} + \frac{1}{m} < x - \varepsilon$. So there are finitely many possible pairs (n, m) with $x - \varepsilon < \frac{1}{n} + \frac{1}{m} < x + \varepsilon$.

Since there are only finitely many elements of $\mathcal{A}$ inside $(x-\varepsilon, x+\varepsilon)$ we can find $k \in \mathbb{N}$ so that $(x-\frac{\varepsilon}{k}, x+\frac{\varepsilon}{k})$ contains no element of $\mathcal{A}$ except possibly x itself. This proves that x is not a limit point of $\mathcal{A}$.

□

Problem 7.

Proof. (a) The points in E^0 are interior points *of* E, to show that E^0 is open we need to show that they are interior points *of* E^0. Given $x \in E^0$, by definition, there exist a open ball $x \in B_r(x) \subset E$. Consider an open ball $B_{r/3}(x) \subset B_r(x)$. Then for any point $y \in B_{r/3}(x)$, $B_{r/3}(y) \subset B_r(x) \subset E$, so $y \in E^0$. Then $B_{r/3}(x)$ is an open ball in E^0. So x is an interior point of E^0.

(b)If $E = E^0$, from (a) we know that E is open. Conversely, if E is open, all points in E are interior points, so $E \subset E^0$. From $E^0 \subset E$ we have $E = E^0$.

(c)Since G is open, so for any point $g \in G$, we have an open ball $B_r(g) \subset G \subset E$. So g is also an interior point of E. Then $G \subset E^0$. □