18.100B Problem Set 1 Solutions

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1) The proof is by contradiction. Assume $\exists r \in \mathbb{Q}$ such that $r^2 = 12$. Then we may write r as $\frac{a}{b}$ with $a, b \in \mathbb{Z}$ and we can assume that a and b have no common factors. Then

$$12 = r^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2},$$

so $12b^2 = a^2$.

Notice that 3 divides $12b^2$ and hence 3 divides a^2. It follows that 3 has to divide a (one way to see this: every integer can be written as either $3n$, $3n+1$, or $3n+2$ for some integer n. If you square these three choices, only the first one gives you a multiple of three.)

Let $a = 3k$, for $k \in \mathbb{Z}$. Then substitution yields $12b^2 = (3k)^2 = 9k^2$, so dividing by 3 we have $4b^2 = 3k^2$, so 3 divides $4b^2$ and hence 3 divides b^2. Just as for a, this implies that b has to divide b. But then a and b share the common factor of 3, which contradicts our choice of representation of r. So there is no rational number whose square is 12.

2) $S \subseteq \mathbb{R}$, $S \neq \emptyset$, and $u = \sup S$. Given any $n \in \mathbb{N}$, $\forall s \in S$, $s \leq u < u + \dfrac{1}{n}$, so $u + \dfrac{1}{n}$ is an upper bound for S. Assume $u - \dfrac{1}{n}$ is also an upper bound for S. Since $u - \dfrac{1}{n} < u$, u would not be the least upper bound for S, which is a contradiction. Therefore $u - \dfrac{1}{n}$ is not an upper bound for S.

3) Recall that a subset of the real numbers, $A \subseteq \mathbb{R}$, is bounded if there are real numbers a and a' such that

$$t \in A \implies a' \leq t \leq a.$$

Since $A, B \subseteq \mathbb{R}$ are bounded, they have upper bounds a and b respectively, and lower bounds $a\prime$ and $b\prime$. Let $\alpha = \max(a, b)$ and $\beta = \min(a\prime, b\prime)$. Clearly,

$$t \in A \implies \beta \leq a' \leq t \leq a \leq \alpha$$

$$t \in B \implies \beta \leq b' \leq t \leq b \leq \alpha,$$

hence any $t \in A \cup B$ satisfies $\beta \leq t \leq \alpha$ and $A \cup B$ is bounded.

Notice that, in particular, this shows that $\max\{\sup A, \sup B\}$ is an upper bound for $A \cup B$, so we only have to show that it is the *least* upper bound. Suppose $\gamma < \max\{\sup A, \sup B\}$. Then without loss of generality, $\gamma < \sup A$. By definition of supremum, γ is not an upper bound of A, so $\exists a \in A$ with $\gamma < a$. But $a \in A \Rightarrow a \in A \cup B$, so γ is not an upper bound of $A \cup B$. Therefore $\max\{\sup A, \sup B\} = \sup A \cup B$.

4) Start by noting that, if $n, m \in \mathbb{N}$ then $b^n b^m = b^{n+m}$ from which it follows that $b^n b^m = b^{n+m}$ for $n, m \in \mathbb{Z}$ (why?). Similarly, you can show that $b^{nm} = (b^n)^m$ for $n, m \in \mathbb{Z}$. Recall that, if $x > 0$, then $x^{\frac{1}{n}}$ is defined to be the *unique* positive real number such that $\left(x^{\frac{1}{n}}\right)^n = x$.

a) We have that $m/n = p/q$ so $mq = pn$. Notice that $\left((b^m)^{\frac{1}{n}}\right)^{nq} = (b^m)^q = b^{mq}$ and that $\left((b^p)^{\frac{1}{q}}\right)^{nq} = (b^p)^n = b^{pn}$, which is also equal to b^{mq}. But we know that there is a *unique* real number y satisfying $y^{nq} = b^{mq}$ hence the two numbers we started with have to be equal, i.e.,

$$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}.$$

Notice that if this equality didn't hold, then we could not make sense of the symbol b^r for $r \in \mathbb{Q}$, because the value would change if we wrote the same number r in two different ways.

b) Let $r, s \in \mathbb{Q}$ with $r = \dfrac{m}{n}$ and $s = \dfrac{p}{q}$. Since nq is an integer we know that

$$(b^r b^s)^{nq} = (b^r)^{nq} (b^s)^{nq}$$

but $(b^r)^{nq} = \left((b^m)^{\frac{1}{n}}\right)^{nq} = (b^m)^q = b^{mq}$ and similarly $(b^s)^{nq} = b^{np}$. Since mq and np are integers we can conclude

$$(b^r b^s)^{nq} = b^{mq} b^{np} = b^{mq+np}.$$

But there is a unique positive real number, y, such that $y^{nq} = b^{mq+np}$, so we know that

$$b^r b^s = \left(b^{mq+np}\right)^{\frac{1}{nq}} = b^{\frac{mq+np}{nq}} = b^{\frac{m}{n}+\frac{p}{q}} = b^{r+s}.$$

c) Now with $b > 1$, given $r, s \in \mathbb{Q}$, $s \le r$ we want to show $b^s \le b^r$. Let $r - s = \frac{m}{n}$, $0 < n$, $0 \le m$ since $s \le r$. Then $b^{r-s} = (b^m)^{\frac{1}{n}}$, and it is easy to see that $1 \le b^m$, since $0 \le m$ and $1 < b$. Thus a positive power of b^{r-s} is greater than or equal to 1, which implies $1 \le b^{r-s}$. Multiplying by b^s gives $b^s \le b^{r-s} b^s = b^{(r-s)+s} = b^r$, so $b^s \le b^r$. Hence for any $b^s \in B(r)$, $s \le r \Rightarrow b^s \le b^r$, so b^r is an upper bound for $B(r)$. Since $b^r \in B(r)$, b^r must be the least upper bound, so $b^r = \sup B(r)$.

d) So let $x, y \in \mathbb{R}$. If $r, s \in \mathbb{Q}$ are such that $r \le x$, $s \le y$, then $r + s \le x + y$ so $b^{r+s} \in B(x+y)$ and $b^r b^s \le b^{x+y}$. Keeping s fixed, notice that for any $r \le x$ we have

$$b^r \le \frac{b^{x+y}}{b^s},$$

thus $\frac{b^{x+y}}{b^s}$ is an upper bound for $B(x)$ which implies $b^x \le \frac{b^{x+y}}{b^s}$. We rearrange this to

$$b^s \le \frac{b^{x+y}}{b^x}$$

and conclude that $b^y \le \frac{b^{x+y}}{b^x}$ or $b^x b^y \le b^{x+y}$.
Suppose the inequality is strict. Then $\exists t \in \mathbb{Q}$, $t < x + y$, such that $b^x b^y < b^t$ [1]. We will find $r, s \in \mathbb{Q}$, with $r \le x$, $s \le y$ and $t < r + s < x + y$. First, find $N \in \mathbb{N}$ so that $N(x + y - t) > 1$, then find $r \in \mathbb{Q}$ so that $x - \frac{1}{2N} < r < x$ and $s \in \mathbb{Q}$ such that $y - \frac{1}{2N} < s < y$ (the existence of N, r, s follow from the Archimedean property of $\mathbb{R}$ as shown in class). Now, notice that

$$N(x + y - t) > 1 \implies t < x + y - \frac{1}{N},$$

$$x - \frac{1}{2N} < r < x \text{ and } y - \frac{1}{2N} < s < y \implies x + y - \frac{1}{N} < r + s < x + y$$

hence we have $t < r + s < x + y$ just like we wanted.

[1] This is true even if $x + y \in \mathbb{Q}$, notice that $\sup B(x+y) = \sup \left\{b^t : t \in \mathbb{Q}, t < x + y\right\}$

But now we have

$$b^x b^y < b^t < b^{r+s} = b^r b^s$$

which is a contradiction because, since $r < x$ and $s < y$, we have $b^r < b^x$ and $b^s < b^y$! [2]

5) We know that in any ordered field, squares are greater than or equal to zero. Since $i^2 = -1$, this means that $0 \leq -1$. But then $1 = 0 + 1 \leq -1 + 1 = 0 \leq 1$ which implies $0 = 1$, a contradiction!

6) I'll write $\ll$ for this relation on $\mathbb{C}$ to distinguish it from the normal order on $\mathbb{R}$. To show that $\ll$ is an order on $\mathbb{C}$, we must show both transitivity and totality (or given $x, y \in \mathbb{C}$, exactly one of the following is true: $x \ll y$, $y \ll x$, or $x = y$). First for transitivity, let $x, y, z \in \mathbb{C}$, $x = a + bi$, $y = c + di$, $z = e + fi$ such that $x \ll y \ll z$. Therefore $a \leq c \leq e$, so $a \leq e$ by the transitivity of the order on $\mathbb{R}$. If $a < e$, then $x \ll z$, so we are done. If $a = e$, then $a = c = e$ so we have from the definition of $\ll$ that $b < d < f$, so once again by the transitivity of the order on $\mathbb{R}$, $b < f$. Now $a = e$ and $b < f \Rightarrow x \ll z$, so we have shown transitivity.

Now to show totality. Consider $x, y \in \mathbb{C}$, $x = a + bi$, $y = c + di$. Without loss of generality, let $a \leq c$. Suppose $a = c$. Then $b < d \Leftrightarrow x \ll y$, $b > d \Leftrightarrow y \ll x$, and $b = d \Leftrightarrow x = y$, so by the totality of the order on $\mathbb{R}$, we have the totality of $\ll$ on $\mathbb{C}$ in the case of $a = c$. Suppose instead that $a < c$. Then we know $x \ll y$, and it is not the case that $y \ll x$ or $x = y$, so we have totality in this case as well. Thus we have proven that $\ll$ is an order on $\mathbb{C}$.

This order does not have the least-upper-bound property. Consider the set of complex numbers with real part less than or equal to zero:

$$S = \{a + bi : a \leq 0, b \in \mathbb{R}\}.$$

S is bounded above, for instance by the number 1, but it is not possible for any number $z = a+bi$ to be the supremum of S. If $a \leq 0$, then $a + bi \ll a + (b + 1)i \in S$, so $a + bi$ is not an upper bound for S. If $a > 0$, then $a + (b - 1)i \ll a + bi$, and $a + (b - 1)i$ is also an upper bound for S, so $a + bi$ is not the least upper bound. Therefore S has no least upper bound, even though it is bounded above.

7) $x, y \in \mathbb{R}^k$, so let $x = (a_1, a_2, ..., a_k)$, $y = (b_1, b_2, ..., b_k)$. Then

$$|x + y|^2 + |x - y|^2 = \sum_{i=1}^{k}(a_i + b_i)^2 + \sum_{j=1}^{k}(a_j - b_j)^2 = \sum_{i=1}^{k}\Big[(a_i + b_i)^2 + (a_i - b_i)^2\Big]$$

$$= \sum_{i=1}^{k}(a_i^2 + 2a_ib_i + b_i^2 + a_i^2 - 2a_ib_i + b_i^2) = \sum_{i=1}^{k}(2a_i^2 + 2b_i^2) = 2(|x|)^2 + 2(|y|)^2.$$

The geometric interpretation comes from looking at the parallelogram whose vertices are the points 0, x, $x + y$ and y. Then the equation states that the sum of the squares of the lengths of the two diagonals (the vectors $x + y$ and $x - y$) is the same as the sum of the squares of the lengths of the four sides.

[2] A different proof of $b^{x+y} \leq b^x b^y$ could start by justifying $b^z = \inf\{b^r : r \in \mathbb{Q}, r \geq z\}$ and then proceeding as in the proof of $b^x b^y \leq b^{x+y}$.