SOLUTIONS TO PS 10

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Solution/Proof of Problem 1. *From the definition, we can find that*

$$f_n(t) = (\frac{2}{3})^n f_0(t) + \sum_{k=0}^{n-1} (\frac{2}{3})^k.$$

Notice that $\lim_{n\to\infty} \sum_{k=0}^{n-1} (\frac{2}{3})^k = \frac{1}{1-2/3} = 3$ *and since* $|f_0(t)| = |\sin t| \leq 1$ *we have* $|f_n - 3| \leqslant |(\frac{2}{3})^n| + |\sum_{k=0}^{n-1} (\frac{2}{3})^k - 3|$ *so we have* $\forall \epsilon > 0$,

- $\exists N_1$ *s.t.* $\forall n > N_1$, $(\frac{2}{3})^n < \frac{\epsilon}{3}$;
- $\exists N_2$ *s.t.* $\forall n > N_1$, $|\sum_{k=0}^{n-1} (\frac{2}{3})^k - 3| < \frac{\epsilon}{3}$.

So take $N = \max\{N_1, N_2\}$, *and we have* $\forall n > N$,

$$|f_n - 3| \leqslant |(\frac{2}{3})^n| + |\sum_{k=0}^{n-1} (\frac{2}{3})^k - 3| \leqslant \frac{2\epsilon}{3} < \epsilon.$$

So $f_n \to 3$ *uniformly on* $\mathbb{R}$.

In general, since $f_n(x) = T^n(f_0(x))$, T *is a contraction, and the fixed point of* T *is* 3, *we always have pointwise convergence of* f_n *to* 3. *However, from the argument above we see that this is uniform convergence if and only if the function* f_0 *is bounded. Thus for* $f_0(t) = t^2$, *the convergence is uniform on any bounded subset of* $\mathbb{R}$, *but not on all of* $\mathbb{R}$.

Solution/Proof of Problem 2. *Since* $t \geq 0$, *we have* $0 \leq \phi(t) \leq \frac{t}{2+t} \leq \frac{t}{2}$. *So we have*

$$0 \leq f_n(t) = \phi(f_{n-1}(t)) \leq \frac{1}{2} f_n(t) \leq \cdots \leq \frac{1}{2^n} f_0(t) = \frac{1}{2^n}\phi(t) \leq \frac{1}{2^n}\frac{t}{2+t} \leq \frac{1}{2^n}.$$

From the convergence of $\sum \frac{1}{2^n}$, *we have* $\sum_{n=0}^{K} f_n(t) \to F(t)$ *uniformly, since each partial sum* $\sum_{n=0}^{K} f_n(t)$ *is continuous, this implies that* F *is continuous.*

Solution/Proof of Problem 3. *Since differentiability is a local property, we only need to prove that* f *is differentiable on every subset* $(-s, s) \subset \mathbb{R}$.

We have

$$\frac{d}{dt}\sin^2(\frac{t}{k}) = \frac{2}{k}\sin(\frac{t}{k})\cos(\frac{t}{k}) = \frac{1}{k}\sin(\frac{2t}{k}),$$

so if $F_n(t) = \sum_{k=1}^{n} \sin^2(\frac{t}{k})$, *then*

$$\frac{d}{dt}F_n = \sum_{k=1}^{n} \frac{1}{k}\sin(\frac{2t}{k}).$$

We can use $|\sin x| \le |x|$ to see that $F_n'(t)$ is uniformly Cauchy; indeed, whenever $n < m$ we have

$$\begin{aligned}\|F_n' - F_m'\| &= \sup_{t\in[-s,s]} |\sum_{k=n}^{m} \frac{1}{k}\sin(\frac{2t}{k})| \le \sup_{t\in[-s,s]} \sum_{k=n}^{m} |\frac{1}{k}\sin(\frac{2t}{k})| \\ &\le \sup_{t\in[-s,s]} \sum_{k=n}^{m} \frac{1}{k}\left(\frac{2|t|}{k}\right) = 2s\sum_{k=n}^{m}\frac{1}{k^2}\end{aligned}$$

and since $\sum \frac{1}{k^2}$ converges, we can make this last sum as small as we like. It follows that $F_n'(t)$ converges uniformly, it's also clear that $F_n(0) \to 0$. From Theorem 7.17, we know that $F_n(t)$ converges to a function $F(t)$ such that $F'(t)$ exists and $F'(t) = \lim_{n\to\infty} F_n'(t)$. So we get the conclusion.

Solution/Proof of Problem 4. *Since $f_n \to f$ uniformly, and f_n are continuous, so is f. Now for any $\epsilon > 0$, we have*

- $\exists N_1$, *s.t.* $\forall n > N_1$, *and* $\forall x \in E$, $|f(x) - f_n(x)| < \frac{\epsilon}{3}$;
- $\exists \delta > 0$, *s.t.* $\forall |x-y| < \delta$, $|f(y) - f(x)| < \frac{\epsilon}{3}$;
- $\exists N_2$, *s.t.* $\forall n > N_2$, $|x - x_n| < \delta$.

So we have for $n > \max\{N_1, N_2\}$,

$$|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \le \frac{2\epsilon}{3} \le \epsilon.$$

So we get the conclusion.

The converse can be formulated different ways. Here's one that's true: If (f_n) is a sequence of continuous funtions that converge pointwise to a function f on a compact set E, *and $\lim_{y\to x} f(y)$ always exists, then*

$$f_n \to f \text{ uniformly} \iff \lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} f(x_n) \text{ whenever } (x_n) \text{ converges.}$$

The proof of $\to$ is above, to prove $\leftarrow$ assume that f_n does not converge to f uniformly. This implies that

$$\begin{aligned} &\text{for some } \varepsilon_0 > 0 \text{ and for every } N \in \mathbb{N} \\ &\qquad \text{there exists } n > N \text{ such that } \|f_n(x_n) - f(x_n)\| > \varepsilon_0 \\ \iff &\text{for some } \varepsilon_0 > 0 \text{ and for every } N \in \mathbb{N} \\ &\qquad \text{there exists } n > N \text{ and } y_n \in E \text{ such that } |f_n(y_n) - f(y_n)| > \varepsilon_0 \end{aligned}$$

Since E is compact, the sequence (y_n) has a convergent subsequence, which we denote (x_n). Say that $\lim_{n\to\infty} f(x_n) = L$ and find $N' \in \mathbb{N}$ such that $n > N'$ implies $|f(x_n) - L| < \varepsilon_0/2$. Then, for any $n > N'$ we have $|f_n(x_n) - L| > \varepsilon_0/2$ and hence

$$\lim_{n\to\infty} f_n(x_n) \neq \lim_{n\to\infty} f(x_n),$$

which proves the converse.

Notice that if we do not require the original sequence to be continuous, then the converse is not true. Take $E = \mathbb{R}$. Consider a sequence of functions

$$f_n(x) = \begin{cases} 0 & x \in (-n, n) \\ 1 & \text{otherwise} \end{cases}$$

Then f_n converge to 0 pointwise and f_n does not converge uniformly to 0. But we can easily see that for any convergent sequence $\{x_n\}$, $f_n(x_n) \to 0$.

Solution/Proof of Problem 5. *Form condition (b), we have*

$$0 \le \int_0^\infty f_n(t)dt \le \int_0^\infty e^{-t} = 1.$$

So we have

$$0 \le \lim_{T\to\infty} \int_T^\infty f_n(t)dt \le \lim_{T\to\infty} \int_T^\infty e^{-t} = 0.$$

So for $\frac{\varepsilon}{3} > 0$, $\exists S$ *s.t.* $\forall n$

$$0 \le \int_S^\infty f_n(t)dt \le \int_S^\infty e^{-t} \le \frac{\varepsilon}{3}.$$

On the other hand, from condition (a), we have for $\frac{\varepsilon}{3S} > 0$, $\exists N$ *s.t.* $\forall n > N$

$$\int_0^S f_n(t)dt \le \int_0^S \frac{\varepsilon}{3S}dt = \frac{\varepsilon}{3}.$$

So we have for $\varepsilon > 0$, $\exists N$ *s.t.* $\forall n > N$

$$\int_0^\infty f_n(t)dt \le \int_0^S f_n(t)dt + \int_S^\infty f_n(t)dt \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.$$

So $\lim\limits_{n\to\infty} \int_0^\infty f_n(t)dt = 0$.

Condition (b) is necessary. In fact, we can consider $f_n(t) = \frac{t}{n}$*, which satisfies condition (a). It is clear that the conclusion does not hold.*

Solution/Proof of Problem 6. *Since* $\{f_n\}$ *is equicontinuous, so for any* $\frac{\varepsilon}{3} > 0$, $\exists \delta > 0$ *s.t. for any two points* $x, y \in K$*, if* $|x - y| < \delta$*, then* $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$.

Now consider an open covering $K = \bigcup_{x\in K} D_\delta(x)$ *where* $D_\delta(x)$ *is a disc with center* x *and radius* δ*. Since* K *is compact, we can find finite disc to cover* K*. Let* $K = \bigcup_{i=1}^n D_\delta(x_i)$, .

For any $x \in K$*, we have* $x \in D_\delta(x_i)$ *for some* x_i*. So* $|f_n(x) - f_n(x_i)| < \frac{\varepsilon}{3}$.

For each i, $f_n(x_i) \to f(x_i)$*, we have for* $\frac{\varepsilon}{3} > 0$, $\exists N_i > 0$, *s.t.* $\forall n > N_i$, $|f_n(x_i) - f(x_i)| < \frac{\varepsilon}{3}$*. Let* $N = \max_i N_i$*, so* $\forall n > N$, $|f_n(x_i) - f(x_i)| < \frac{\varepsilon}{3}$ *for any* i.

So $\forall m, n > N$

$$|f_m(x) - f_n(x)| \le |f_m(x) - f_m(x_i)| + |f_m(x_i) - f_n(x_i)| + |f_n(x_i) - f_n(x)| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \le \varepsilon.$$

So $f_n \to f$ *uniformly.*

Solution/Proof of Problem 7. *Since* f_n' *is uniformly bounded, so there exists* M *s.t.* $|f_n'(x)| \le M, \forall x, n$.

For any $x \le y$*, by MVT, we have* $\exists \xi \in [x, y]$ *s.t.*

$$|f_n(x) - f_n(y)| = |f_n'(\xi)(x - y)| \le M|x - y|.$$

So for any $\epsilon > 0$, $\exists \delta = \epsilon/M > 0$ *s.t.* $\forall x, y, |x - y| < \delta$, $|f_n(x) - f_n(y)| < \epsilon$.

So f_n *is equicontinuous. Then from Arzela-Ascoli theorem, we got the conclusion.*

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