18.100B Problem Set 1

Due Friday September 15, 2006 by 3 PM

Problems.

1) (10 pts) Prove that there is no rational number whose square is 12.

2) (10 pts) Let S be a non-empty subset of the real numbers, bounded above. Show that if $u = \sup S$, then for every natural number n, the number $u - \frac{1}{n}$ is not an upper bound of S, but the number $u + \frac{1}{n}$ is an upper bound of S.

3) (10 pts) Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded subset of $\mathbb{R}$. Show that
$$\sup A \cup B = \max\{\sup A, \sup B\}.$$

4) (20 pts) Fix $b > 1$.
   a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that
   $$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}.$$
   Hence it makes sense to define $b^r = (b^m)^{\frac{1}{n}}$. (How could it have failed to make sense?)
   b) Proce that $b^{r+s} = b^r b^s$ if r, s are rational.
   c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that
   $$b^r = \sup B(r)$$
   when r is rational. Hence it makes sense to *define*
   $$b^x := \sup B(x)$$
   for every real x.
   d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

5) (10 pts) Prove that no order can be defined in the complex field that turns it into an ordered field.
(*Hint:* -1 is a square.)

6) (10 pts) Suppose $z = a + bi$, $w = c + di$. Define
$$z < w \text{ if } a < c \text{ or } a = c, b < d$$
Prove that this turns the set of all complex numbers into an ordered set. (This is known as a dictionary order, or lexicographic order.) Does this ordered set have the least-upper-bound property?

7) (10 pts) Prove that
$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$$
if $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

Extra problems:

1) (Another argument showing that $\sqrt{2} \notin \mathbb{Q}$)
Show that, if $n^2 = 2m^2$, then
$$(2m - n)^2 = 2(n - m)^2.$$
Deduce that, if n and m are strictly positive integers with $n^2 = 2m^2$, we can find strictly positive integers n', m' with $(n')^2 = 2(m')^2$ and $n' < n$. Conclude that the equation $n^2 = 2m^2$ has no non-zero integer solutions.

2) Show that the square root of an integer is either an integer or irrational.
(*Hint*: Every integer has a unique (up to order) factorization into a product of prime numbers, you can use this to show that if n is an integer and a prime p divides n^2, then p divides n.)