7 Operators

This chapter is an extended example of an analogy. In the last chapter, the analogy was often between higher- and lower-dimensional versions of a problem. Here it is between operators and numbers.

7.1 Derivative operator

Here is a differntial equation for the motion of a damped spring, in a suitable system of units:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = 0,$$

where x is dimensionless position, and t is dimensionless time. Imagine x as the amplitude divided by the initial amplitude; and t as the time multiplied by the frequency (so it is radians of oscillation). The dx/dt term represents the friction, and its plus sign indicates that friction dissipates the system's energy. A useful shorthand for the d/dt is the operator D. It is an operator because it operates on an object – here a function – and returns another object. Using D, the spring's equation becomes

$$D^2x(t) + 3Dx(t) + x(t) = 0.$$

The tricky step is replacing d^2x/dt^2 by D^2x, as follows:

$$D^2x = D(Dx) = D\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}.$$

The analogy comes in solving the equation. Pretend that D is a number, and do to it what you would do with numbers. For example, factor the equation. First, factor out the $x(t)$ or x, then factor the polynomial in D:

$$(D^2 + 3D + 1)x = (D + 2)(D + 1)x = 0.$$

This equation is satisfied if either $(D + 1)x = 0$ or $(D + 2)x = 0$. The first equation written in normal form, becomes

$$(D + 1)x = \frac{dx}{dt} + x = 0,$$

or $x = e^{-t}$ (give or take a constant). The second equation becomes

$$(D + 2)x = \frac{dx}{dt} + 2x = 0,$$

or $x = e^{-2t}$. So the equation has two solutions: $x = e^{-t}$ or e^{-2t}.

7.2 Fun with derivatives

The example above introduced D and its square, D^2, the second derivative. You can do more with the operator D. You can cube it, take its logarithm, its reciprocal, and even its exponential. Let's look at the exponential e^D. It has a power series:

$$e^D = 1 + D + \frac{1}{2}D^2 + \frac{1}{6}D^3 + \cdots.$$

That's a new operator. Let's see what it does by letting it operating on a few functions. For example, apply it to $x = t$:

$$(1 + D + D^2/2 + \cdots)t = t + 1 + 0 = t + 1.$$

And to $x = t^2$:

$$(1 + D + D^2/2 + D^3/6 + \cdots)t^2 = t^2 + 2t + 1 + 0 = (t + 1)^2.$$

And to $x = t^3$:

$$(1 + D + D^2/2 + D^3/6 + D^4/24 + \cdots)t^3 = t^3 + 3t^2 + 3t + 1 + 0 = (t + 1)^3.$$

It seems like, from these simple functions (extreme cases again), that $e^D x(t) = x(t + 1)$. You can show that for any power $x = t^n$, that

$$e^D t^n = (t + 1)^n.$$

Since any function can, pretty much, be written as a power series, and e^D is a linear operator, it acts the same on any function, not just on the powers.

So e^D is the successor operator: It turns the function $x(t)$ into the function $x(t+1)$.

Now that we know how to represent the successor operator in terms of derivatives, let's give it a name, S, and use that abstraction in finding sums.

7.3 Summation

Suppose you have a function $f(n)$ and you want to find the sum $\sum f(k)$. Never mind the limits for now. It's a new function of n, so summation, like integration, takes a function and produces another function. It is an operator, σ. Let's figure out how to represent it in terms of familiar operators. To keep it all straight, let's get the limits right. Let's define it this way:

$$F(n) = (\sum f)(n) = \sum_{-\infty}^{n} f(k).$$

So $f(n)$ goes into the maw of the summation operator and comes out as $F(n)$. Look at $SF(n)$. On the one hand, it is $F(n+1)$, since that's what S does. On the other hand, S is, by analogy, just a number, so let's swap it inside the definition of $F(n)$:

$$SF(n) = (\sum Sf)(n) = \sum_{-\infty}^{n} f(k+1).$$

The sum on the right is $F(n) + f(n+1)$, so

$$SF(n) - F(n) = f(n+1).$$

Now factor the $F(n)$ out, and replace it by σf:

$$((S-1)\sigma f)(n) = f(n+1).$$

So $(S-1)\sigma = S$, which is an implicit equation for the operator σ in terms of S. Now let's solve it:

$$\sigma = \frac{S}{S-1} = \frac{1}{1-S^{-1}}.$$

Since $S = e^D$, this becomes

$$\sigma = \frac{1}{1-e^{-D}}.$$

Again, remember that for our purposes D is just a number, so find the power series of the function on the right:

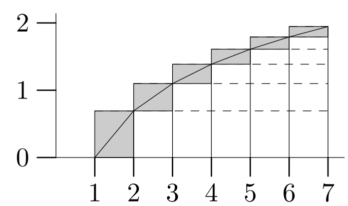
$$\sigma = D^{-1} + \frac{1}{2} + \frac{1}{12}D - \frac{1}{720}D^3 + \cdots .$$

The coefficients do not have an obvious pattern. But they are the Bernoulli numbers. Let's look at the terms one by one to see what the mean. First is D^{-1}, which is the inverse of D. Since D is the derivative operator, its inverse is the integral operator. So the first approximation to the sum is the integral – what we know from first-year calculus.

The first correction is 1/2. Are we supposed to add 1/2 to the integral, no matter what function we are summing? That interpretation cannot be right. And it isn't. The 1/2 is one piece of an operator sum that is applied to a function. Take it in slow motion:

$$\sigma f(n) = \int^n f(k)\, dk + \frac{1}{2} f(n) + \cdots .$$

So the first correction is one-half of the final term $f(n)$. That is the result we got with this picture from **Section 4.6**. That picture required approximating the excess as a bunch of triangles, whereas they have a curved edge. The terms after that correct for the curvature.



7.4 Euler sum

As an example, let's use this result to improve the estimate for Euler's famous sum

$$\sum_{1}^{\infty} n^{-2}.$$

The first term in the the operator sum is 1, the result of integrating n^{-2} from 1 to ∞. The second term is 1/2, the result of $f(1)/2$. The third term is 1/6, the result of $D/12$ applied to n^{-2}. So:

$$\sum_{1}^{\infty} n^{-2} \approx 1 + \frac{1}{2} + \frac{1}{6} = 1.666\ldots$$

The true value is 1.644 . . ., so our approximation is in error by about 1%. The fourth term gives a correction of $-1/30$. So the four-term approximation is 1.633 . . ., an excellent approximation using only four terms!

7.5 Conclusion

I hope that you've enjoyed this extended application of analogy, and more generally, this rough-and-ready approach to mathematics.