2 Extreme cases

The next item for your toolbox is the method of **extreme cases**. You can use it to check results and even to guess them, as the following examples illustrate.

2.1 Fencepost errors

Fencepost errors are the most common programming mistake. An index loops over one too many or too few items, or an array is allocated one too few memory locations – leading to a buffer overrun and insecure programs. Since programs are a form of mathematics, fencepost errors occur in mathematics as well. The technique of extreme cases helps you find and fix these errors and deduce correct results instead.

Here is the sum of the first n odd integers:

$$S = \underbrace{1 + 3 + 5 + \cdots}_{n \text{ terms}}$$

Odd numbers are of the form $2k+1$ or $2k-1$. Quickly answer this question:

Is the last term $2n+1$ or $2n-1$?

For a general n, the answer is not obvious. You can figure it out, but it is easy to make an algebra mistake and be off by one term, which is the difference between $2n-1$ and $2n+1$. An extreme case settles the question. Here is the recipe for this technique:

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1. Pick an extreme value of n, one where the last term in the sum is easy to determine.
2. For that n, determine the last term.
3. See which prediction, $2n-1$ or $2n+1$ (or perhaps neither), is consistent with this last term.

The most extreme value of n is 0. Since n is the number of terms, however, the meaning of $n = 0$ is obscure. The next most extreme case is $n = 1$. With only one term, the final (and also first) term is 1, which is $2n - 1$. So the final term, in general, should be $2n - 1$. Thus the sum is

$$S = 1 + 3 + 5 + \cdots + 2n - 1.$$

Using sigma notation, it is

$$S = \sum_{k=0}^{n-1} (2k + 1).$$

This quick example gives the recipe for extreme-cases reasoning; as a side benefit, it may help you spot bugs in your programs. The sum itself has an elegant picture, which you learn in **Section 4.1** in the chapter on pictorial proofs. The rest of this chapter applies the extreme-cases recipe to successively more elaborate problems.

2.2 Integrals

An integral from the **Chapter 1**, on dimensions, can illustrate extreme cases as well as dimensions. Which of these results is correct:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \begin{cases} \sqrt{\alpha\pi} \\ \text{or} \\ \sqrt{\dfrac{\pi}{\alpha}} \end{cases} \quad ?$$

Dimensional analysis answered this question, but forget that knowledge for the moment so that you can practice a new technique.

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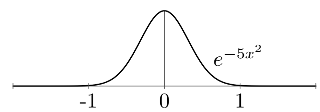
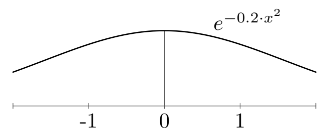
You can make the correct choice by by looking at the integrand $e^{-\alpha x^2}$ in the two extremes $\alpha \to \infty$ and $\alpha \to 0$. As α becomes large, the exponent $-\alpha x^2$ becomes large and negative even when x is only slightly greater than zero. The exponential of a large negative number is nearly zero, so the bell curve narrows, and its area shrinks. As $\alpha \to \infty$, the area and therefore the integral should shrink to zero. The first option, $\sqrt{\alpha\pi}$, instead goes to infinity. It must be wrong. The second option, $\sqrt{\pi/\alpha}$, goes to infinity and could be correct.



The complementary test is $\alpha \to 0$. The function flattens to the horizontal line $y = 1$; its integral over an infinite range is infinity. The first choice, $\sqrt{\pi\alpha}$, fails this test because instead it goes to zero as $\alpha \to 0$. The second option, $\sqrt{\pi/\alpha}$, goes to infinity and passes the test. So the second option passes both tests, and the first option fails both tests. This increases my confidence in $\sqrt{\pi/\alpha}$ while decreasing it, nearly to zero, in $\sqrt{\pi\alpha}$.



If those were the only choices, *and I knew that one choice was correct*, I would choose $\sqrt{\pi/\alpha}$. However, if the joker who wrote the problem included $\sqrt{2/\alpha}$ among the choices, then I need a third test to distinguish between $\sqrt{2/\alpha}$ and $\sqrt{\pi/\alpha}$. For this test, use a third extreme case: $\alpha \to 1$. Wait, how is 1 an extreme case? Infinity and zero are extreme, but 1 lies between those two so it cannot be an extreme.

Speaking literally, 1 is a special case rather than an extreme case. So extend the meaning of extreme with poetic license and include special cases. The tool, named in full, would be the 'method of extreme and special cases'. Or, since extreme cases are also special, it could be the 'method of special cases'. The first option, although correct, is unwieldy. The second option, although also sharing the merit of correctness, is cryptic. It does not help you think of special cases, whereas 'extreme cases' does help you: It tells you to look at the extremes. So I prefer to keep the name simple – extreme cases – while reminding myself that extreme cases include special cases like $\alpha \to 1$.

In the $\alpha \to 1$ limit the integral becomes

$$I \equiv \int_{-\infty}^{\infty} e^{-x^2}\,dx,$$

where the $\equiv$ notation means 'is defined to be' (rather than the perhaps more common usage in mathematics for modular arithmetic). It is the Gaussian integral and its value is $\sqrt{\pi}$. The usual trick to compute it is to evaluate the square of the integral:

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$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right) \times \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right).$$

In the second factor, change the integration variable to y, making the product

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\alpha x^2} e^{-\alpha y^2}\,dx\,dy.$$

It looks like the integral has become more complicated, but here comes the magic trick. The exponentials multiply to give $e^{-(x^2+y^2)}$, integrated over all x and y – in other words, over the whole plane. And $e^{-(x^2+y^2)} = e^{-r^2}$. So the square of the Gaussian integral is also, in polar coordinates, the integral $\int_{\text{plane}} e^{-r^2}\,dA$, where dA is the element of area $r\,dr\,d\theta$:

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,\underbrace{r\,dr\,d\theta}_{dA}.$$

This integral is doable because the r contributed by the dA is the derivative, except for a factor of 2, of the r^2 in the exponent:

$$\int e^{-r^2} r\,dr = \frac{1}{2}e^{-r^2} + C,$$

and

$$\int_0^{\infty} e^{-r^2} r\,dr = \frac{1}{2}.$$

The $d\theta$ integral contributes a factor of 2π so $I^2 = 2\pi/2 = \pi$ and the Gaussian integral is its square root:

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

The only choice consistent with all three extreme cases, even with $\sqrt{2/\alpha}$ among them, is

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}.$$

This integral could also be guessed by dimensions, as explained in **Section 1.2**. Indeed dimensions tell you more than extreme cases do. Dimensions refutes $\sqrt{\pi}/\alpha$ or $\sqrt{\pi}/\alpha^2$, whereas both choices pass the three extreme-case tests:

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- $\alpha \to 0$ Both choices correctly limit to ∞.
- $\alpha \to \infty$ Both choices correctly limit to 0.
- $\alpha \to 1$ Both choices correctly limit to $\sqrt{\pi}$.

Extreme cases, however, has the virtue of being quick. You do not need to find the dimensions for x or α (or invent the dimensions), then find the dimensions of dx and of the result. Extreme cases immediately refutes $\sqrt{\pi\alpha}$. The technique's other virtues become apparent in the next problem: how a pendulum's period varies with amplitude.

2.3 Pendulum

In physics courses, the first problem on oscillations is the ideal spring. Its differential equation is

$$m\frac{d^2x}{dt^2} + kx = 0,$$

where k is the spring constant. Dividing by m gives

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

A consequence of this equation, which we derived in **Section 1.4**, is that the oscillation period is independent of the amplitude. That property is characteristic of a so-called simple-harmonic system. The oscillation period is:

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

Before moving on to the pendulum, pause to make a sanity check. To make a sanity check, ask yourself: 'Is each portion of the formula reasonable, or does it come out of left field.' [For the non-Americans, left field is one of the distant reaches of a baseball field, and to come out of left fields means an idea come out of nowhere and surprises everyone with how crazy it is.] One species of sanity checking is to check dimensions. Are the dimensions on both sides correct? In this case they are. The dimensions of spring constant are force per length because $F = kx$, so $[k] = \mathrm{MT}^{-2}$. So the dimensions of $\sqrt{m/k}$ are simply time, which is consistent with being an oscillation period

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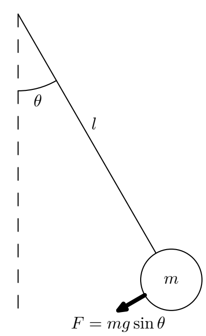
T. [Sorry about the almost-ambiguous notation with T (italic) representing period and T (roman) representing the time dimension.]

Another species of sanity checking is checking extreme cases. Is it reasonable, for example, that m is in the numerator? To decide, check an extreme case of mass. As the mass goes to infinity, the period should go to infinity because the spring has a hard time moving the monstrous mass; and behold, the formula correctly predicts that $T \to \infty$. Is it reasonable that spring constant k is in the denominator? Check an extreme case of k. As $k \to 0$, the spring becomes pathetically weak, and the period should go to infinity. Indeed, the formula predicts that $T \to \infty$. What about the 2π? To find this constant, either solve the differential equation honestly or use a trick invented by Huygens, which I will explain in lecture if you remind me.



Once the spring has been beaten half to death in physics class, the pendulum is sprung on you. We will study how the period of a pendulum depends on its amplitude – on the maximum angle of the swing, normally called θ_0. First, let's derive the differential equation for the pendulum, then deduce properties of its solution without solving it. Just as force fights to linearly accelerate an object with mass, torque fights to angularly accelerate an object with moment of inertia. Compare the following formulas:

$$\text{force} = \text{mass} \times \text{linear acceleration},$$
$$\text{torque} = \text{moment of inertia} \times \text{angular acceleration}.$$

The first formula is Newton's second law, so you can easily remember it. The second formula follows from the first by analogy, which is the technique of **Chapter 6**. Torque is like force; moment of inertia is like mass; and angular acceleration is like linear acceleration.

The moment of inertia of the bob is $I = ml^2$, and angular acceleration is $\alpha \equiv d^2\theta/dt^2$ (again using $\equiv$ to mean 'is defined to be'). The tangential force trying to restore the pendulum bob to the vertical position is $F = mg\sin\theta$. Or is it $mg\cos\theta$? Decide using extreme cases. As $\theta \to 0$, the pendulum becomes directly vertical hanging downward, and the tangential force F goes to zero. Since $\sin\theta \to 0$ as $\theta \to 0$, the force should contain $\sin\theta$ rather than $\cos\theta$.

The torque, which is the force times the lever arm l, is $Fl = mgl\sin\theta$. Putting all three pieces together:

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$$\underbrace{-mgl\sin\theta}_{\text{torque}} = \underbrace{ml^2}_{I} \times \underbrace{\frac{d^2\theta}{dt^2}}_{\alpha},$$

where the minus sign in the torque reflects that it is a restoring torque. The mass divides out to produce the pendulum differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0.$$

This pendulum equation looks similar to the spring equation

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

Comparing the two equations produces these analogies:

$$\begin{aligned} x &\to \theta, \\ \frac{k}{m} &\to \frac{g}{l}, \\ x &\to \sin\theta. \end{aligned}$$

The first two lines are fine, but the third line contradicts the first one: x cannot map to θ and to $\sin\theta$.

Extreme cases help. Sure, θ and $\sin\theta$ are not identical. However, in the extreme case $\theta \to 0$, which means that the oscillation angle θ also goes to zero, the two alternatives θ and $\sin\theta$ are identical (a picture proof is given in ??), For small amplitudes, in other words, the pendulum is almost a simple-harmonic system, which would have a constant period. By analogy with the spring equation, the pendulum's period is

$$T = 2\pi\sqrt{\frac{l}{g}},$$

because the pendulum differential equation has g/l where the spring differential equation has k/m. This extreme case is further analyzed in **Chapter 3** using the technique of discretization.

In the Gaussian integral with α, one extreme case was $\alpha \to 0$ and another was $\alpha \to \infty$. So try that extreme case here, and see what you can deduce. Not much, since an infinite angle is not informative. However, the idea of a large amplitude is suggestive and helpful. The largest meaningful amplitude – set by the angle of release – is 180° or, in radians, $\theta_0 = \pi$. That angle requires a rod as the pendulum 'string', so that the pendulum does not collapse. Such

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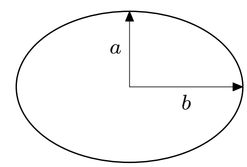
a pendulum balanced at $\theta_0 = \pi$ hangs upside down forever. So $T \to \infty$ when $\theta_0 \to \pi$. Therefore the period should *increase* as amplitude increases. It could decrease initially, for small θ_0, then increase as θ_0 gets near π. That behavior would be nasty. The physical world, at least as a first assumption, does not play such tricks on us.

2.4 Ellipse

Now try extreme cases and dimensions on these candidate formulas for the area A of an ellipse:



a. ab^2

b. $a^2 + b^2$

c. a^3/b

d. $2ab$

e. πab

Let's take them one by one.

- ab^2. This product has dimensions of length cubed rather than length squared, so it flunks the dimensions test and does not even graduate to the extreme-cases tests. But the other choices have correct dimensions and require more work.

- $a^2 + b^2$. Try an extreme ellipse: a super-thin one with $a = 0$. This case satisfies the first step of the recipe:

  > Pick an extreme value where the result is easy to determine without solving the full problem.

  Now do the second step:

  > For that extreme case, determine the result.

  When $a = 0$ the ellipse has zero area no matter what b is. The third step is:

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> Determine the prediction in this extreme case, and compare it with the actual value from the second step.

When $a = 0$, the candidate $A = a^2 + b^2$ becomes $A = b^2$. It can be zero, but alas only when $b = 0$. So the candidate fails this extreme-case test except when $a = 0$ and $b = 0$: a boring case of the ellipse shrinking to a point.

- a^3/b. This candidate passes the thin-ellipse test with $a \to 0$. When $a \to 0$, the predicted and actual areas are zero no matter the value of b. Perhaps the candidate is correct. However, it must pass all tests – and even then it may be wrong. If $a \to 0$ is a reasonable test, then by symmetry $b \to 0$ should also be worth trying. This test pushes the candidate off the stage. When $b \to 0$, which produces an infinitely thin vertical ellipse with zero area, the candidate predicts an infinite area whereas the actual area is zero. Although the candidate passed the first test, it fails the second test.

- $2ab$. This candidate is promising. When $a \to 0$ or $b \to 0$, the actual and predicted areas are zero. So the candidate passes both extreme-case tests. Both $a \to 0$ and $b \to 0$ are literal extreme cases. Speaking figuratively, $a = b$ is also an extreme case. When $a = b$, the candidate predicts that $A = 2a^2$ or, since $a = b$, that $A = 2b^2$. When $a = b$, however, the ellipse is a circle with radius a, and that circle has area πa^2 rather than $2a^2$. So the prediction fails.

- πab. This candidate passes all three tests. Just like $A = 2ab$, it passes $a \to 0$ and $b \to 0$. Unlike $A = 2ab$, this candidate also passes the $a = b$ test (making a circle). With every test that a candidate passes, confidence in it increases. So you can be confident in this candidate. And indeed it is correct.

This example introduces extreme cases in a familiar problem, and one where you have choices to evaluate. We next try a three-dimensional problem and guess the answer from scratch. But before moving on, I review the extreme-case tests and discuss how to choose them. Two natural extremes are $a \to 0$ and $b \to 0$. However, where did the third test $a \to b$ originate, and how would one think of it? The answer is symmetry, a useful trick. Actually it's a method: 'a method is a trick I use twice' (George Polya). Symmetry

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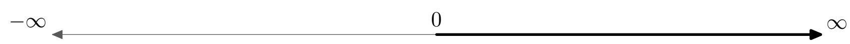
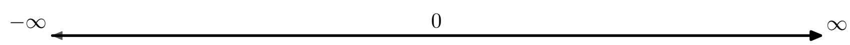
already helped us think of trying $b \to 0$ after we tried $a \to 0$. So the following use of it is the second application. Since a and b are lengths, it is natural to compare them by forming their (dimensionless) ratio a/b. The range of a/b is between 0 and ∞:



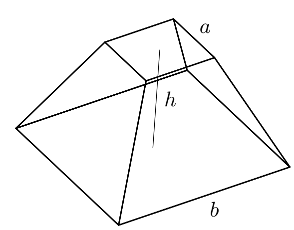
The immediately interesting values in this range are its endpoints 0 and ∞. However, this range is a runt. It is asymmetric, incomplete, and lives on only the right one-half of the real line. To complete the range so that it extends from $-\infty$ to ∞, take the logarithm of a/b. Here are the possible values of $\ln(a/b)$:



The interesting values on this line are again the endpoints, which are $-\infty$ and ∞, but also a new one: the middle point, 0. The interesting values of a/b are 0, 1, and ∞. These points are the three extreme cases for testing the candidate ellipse areas:

$$\begin{aligned} a/b = 0 &\quad\to\quad b = 0, \\ a/b = \infty &\quad\to\quad a = 0, \\ a/b = 1 &\quad\to\quad a = b. \end{aligned}$$

2.5 Truncated pyramid



In the ellipse example, extreme cases helped us evaluate candidates for the area. The next example shows you how to use extreme cases to find a result. Beyond area, the next level of complexity is volume, and the result we look for is the volume of the truncated pyramid formed by slicing off a chunk of the familiar pyramid with a square base. It has therefore a square base and square top that, for simplicity, we assume is parallel to the base. Its height is h, the side length of the base is b, and the side length of the top is a. Guess its volume by finding a formula that meets all the extreme-case tests!

In doing so do not forget the previous technique: dimensions. Any formula must have dimensions of length cubed, so forget about candidate volumes like $V = a^2b^2$ or $V = a^2bh$. But a^2b^2/h would pass the dimensions test.

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What are the extreme cases? The simplest is $h \to 0$, producing a pyramid with zero volume. So a^2b^2/h, although having the correct dimensions, fails because it bogusly produces an infinite volume. Plausible candidates – those producing zero volume – could be ha^2 or h^2a. To choose between those two, think about how the volume must depend on the height. Chop the pyramid into little vertical slivers. When you double the height, you double the height of each sliver, which doubles the volume. So the volume should be proportional to height:

$$V \propto h.$$

A few extreme-cases tests refine this guess. The remaining variables are a and b. The ellipse had only a and b. In the ellipse, a and b are equivalent lengths. Interchanging a and b rotates the ellipse 90° but preserves the same shape and area. For the truncated pyramid, interchanging a and b flips the pyramid 180° but preserves the shape and area. So a and b in the truncated pyramid might have the same interesting extreme cases as do a and b in the ellipse: $a \to 0$, $b \to 0$, and $a \to b$. So let's apply each test in turn, ensuring that the formulas developed in the stepwise process meet all the tests so far investigated.

- $a \to 0$. This limit shrinks the top surface from a square to a point, making the truncated pyramid an ordinary pyramid with volume $hb^2/3$. This formula also passes the $V \propto h$ test. So $V = hb^2/3$ is a reasonable guess for the truncated volume. Continue testing it.

- $b \to 0$. This limit shrinks the bottom surface from a square to a point, producing an upside-down-but-otherwise-ordinary pyramid. The previous candidate $V = hb^2/3$ predicts a zero volume, no matter what a is, so $V = hb^2/3$ cannot be correct. The complementary alternative $V = ha^2/3$ passes the $b \to 0$ test. Great!

  Alas, it fails the first test $a \to 0$. One formula, $V = hb^2/3$, works for $a \to 0$; the other formula, $V = ha^2/3$, works for $b \to 0$. Can a candidate pass both tests? Yes! Add the two half-successful candidates:

  $$V = \frac{1}{3}ha^2 + \frac{1}{3}hb^2 = \frac{1}{3}h(a^2 + b^2).$$

  Two alternatives that also pass both extreme-cases tests, but are not as easy to dream up, are

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$$V = \frac{1}{3}h(a+b)^2.$$

and

$$V = \frac{1}{3}h(a-b)^2.$$

- $a \to b$. In this limit, the pyramid becomes a rectangular prism with height h and base area b^2 (or a^2). So its volume is $V = hb^2$. The hard-won candidate $V = h(a^2 + b^2)/3$, designed to pass the two previous extreme cases, fails this one. Nor do the two alternatives pass. One candidate that does pass is $V = hb^2$. However, it is asymmetric: It treats b specially, which is particularly absurd when $a = b$. What about $V = ha^2$? It treats a specially. What about $V = h(a^2 + b^2)/2$? It is symmetric and passes the $a = b$ test, but it fails the $a \to 0$ and $b \to 0$ tests.

  We need to expand our horizons. One way to do that is to compare the three candidates that passed $a \to 0$ and $b \to 0$:

$$V = \frac{1}{3}h(a^2 + b^2) = \frac{1}{3}h(a^2 + b^2),$$
$$V = \frac{1}{3}h(a + b^2) = \frac{1}{3}h(a^2 + 2ab + b^2),$$
$$V = \frac{1}{3}h(a - b^2) = \frac{1}{3}h(a^2 - 2ab + b^2).$$

The expanded versions share the a^2 and b^2 terms in the parentheses, while differing in the coefficient of the ab term. The freedom to choose that coefficient makes sense. The product ab is 0 in either limit $a \to 0$ or $b \to 0$. So adding any amount of ab in the parentheses will not affect the $a \to 0$ and $b \to 0$ tests. With just the right coefficient of ab, the candidate might also pass the $a = b$ test. Therefore, find the right coefficient n be in

$$V = \frac{1}{3}h(a^2 + nab + b^2).$$

Use the extreme (or special) case $a = b$. Then, the candidate becomes $V = h(2+n)b^2/3$. To make this volume turn into the correct limit hb^2, the numerical factor $(2+n)/3$ should equal 1 meaning that $n = 1$ is the solution:

$$\boxed{V = \frac{1}{3}h(a^2 + ab + b^2).}$$

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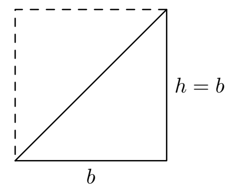
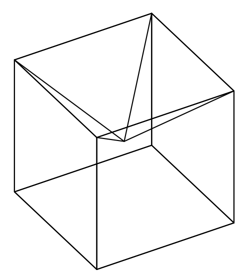
2.6 The magic one-third



You may wonder about the factor of one-third in the volumes of a truncated or regular pyramid. An extreme-case trick explains its origin. First I explain the trick in fewer dimensions: another example of analogy, a technique worthy of its own chapter (**Chapter 6**). Instead of immediately explaining the one-third in the volume of a pyramid, which is a difficult three-dimensional problem, first find the corresponding constant in a two-dimensional problem: the area A of a triangle with base b and height h. Its area is $A \sim bh$. What is the constant? Choose a convenient triangle: perhaps a 45-degree right triangle where $h = b$. Two such triangles form a square with area b^2, so $A = b^2/2$ when $h = b$. The constant in $A \sim bh$ is therefore $1/2$ and $A = bh/2$. Now use the same construction in three dimensions.



What pyramid, when combined with itself perhaps several times, makes a familiar shape? Only the aspect ratio h/b matters in the following discussion. So choose b conveniently, and then choose h to make a pyramid with the clever aspect ratio. The goal shape is suggested by the square pyramid base. Another solid with the same base is a cube. Perhaps several pyramids can combine into a cube of side b. To ease the upcoming arithmetic, I choose $b = 2$. What should h be? To decide, imagine how the cube will be constructed. Each cube has six faces, so six pyramids might make a cube with each pyramid base forming one face of the cube and each pyramid tip facing inwards, meeting in the center of the cube. For the points to meet in the center of the cube, the height must be $h = 1$. So six pyramids with $a = 0$ (meaning that they are not truncated), $b = 2$, and $h = 1$ make a cube with side length 2. The volume of one pyramid is

$$V = \frac{\text{cube volume}}{6} = \frac{8}{6} = \frac{4}{3}.$$

The volume of the pyramid is $V \sim hb^2$, and I choose the missing constant so that the volume is $4/3$. Since $hb^2 = 4$ for these pyramids, the missing constant is $1/3$:

$$V = \frac{1}{3}hb^2 = \frac{4}{3}.$$

So that the general, truncated pyramid agrees with the ordinary pyramid in the limit that $a \to 0$, the constant for the truncated pyramid is also one-third:

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$$V = \frac{1}{3} h(a^2 + ab + b^2).$$

2.7 Drag

The final application of extreme-cases reasoning is to solutions of these nasty nonlinear, coupled, partial-differential equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{v}, \qquad \text{(3 eqns)}$$

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0. \qquad \text{(1 eqn)}$$

The top three equations are the Navier–Stokes equations of fluid mechanics, and the bottom equation is the continuity equation. In the four equations is the answer to the following question:

> When you drop a paper cone (like a coffee filter) and a smaller cone with the same shape, which falls faster?

Solving those equations is a miserable task, which is why we will instead use our two techniques: dimensions and then extreme cases. For the moment, assume that each cone instantly reaches terminal velocity; that approximation is reasonable but we will check it in ?? using the technique of discretization. So we need to find the terminal velocity. It depends on the weight of the cone and on the drag force F resisting the motion.

To find the force, we use dimensions and add a twist to handle problems like this one that have an infinity of dimensionally correct answers. The drag force depends on the object's speed v; on the fluid's density ρ; on its kinematic viscosity ν; and on the object's size r. Now find the dimensions of these quantities and find all dimensionally correct statements that are possible to make about F. Size r has dimensions of L. Terminal velocity v has dimensions of LT^{-1}. Drag force F has dimensions of mass times acceleration, or MLT^{-2}. Density ρ has dimensions of ML^{-3}. The dimensions of viscosity ν are harder. In the problem set, you show that it has dimensions of $\mathrm{L}^2\mathrm{T}^{-1}$. If you look for combinations of ν, ρ, and r, and v that produce dimensions of force, an infinite number of solutions appear, whereas in previous examples using dimensions, only one possibility had the correct dimensions.

Hence the need for a more advanced method to handle the infinite possibilities here. Return to the first principle of dimensions: *you cannot add*

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apples to oranges. The requirement that the sides of an equation match dimensionally is one consequence of the apples-and-oranges principle. Another consequence is that every term in an equation must have the same dimensions. So imagine any true statement about drag force:

$$A + B = C$$

where A, B, and C might be messy combinations of the variables. Then divide each term by A:

$$\frac{A}{A} + \frac{B}{A} = \frac{C}{A}.$$

Because A, B, and C have the same dimensions, each ratio is dimensionless. So you can take any (true) statement about drag force and rewrite it in dimensionless form. No step in this argument depended on the details of drag. It required only that apples must be added to apples. So:

> You can write any true statement about the world in dimensionless form.

Furthermore, you can construct any dimensionless expression using dimensionless groups: products of the variables where the product has no dimensions. Since you can write any true statement in dimensionless form, and can write any dimensionless form using dimensionless groups:

> You can write any true statement about the world using dimensionless groups.

In the problem of free fall, with variables v, g, and h, the dimensionless group is $v/\sqrt{gh}$, perhaps raised to a power. With only one group, the only dimensionless statement has the form:

$$\text{the one group} = \text{dimensionless constant},$$

which results in $v \sim \sqrt{gh}$.

For the drag, what are some dimensionless groups? One group is $F/\rho v^2 r^2$, as you can check by working out its dimensions. A second group is rv/ν. Any other group, it turns out, can be formed from these two groups. With two groups, the most general dimensionless statement is

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$$\text{one group} = f(\text{other group}),$$

where f is a dimensionless function. It has a dimensionless argument and must return a dimensionless value because the left side of the equation is dimensionless. Using $F/\rho v^2 r^2$ as the first group:

$$\frac{F}{\rho v^2 r^2} = f\left(\frac{rv}{\nu}\right).$$

The second group, which is the quantity in the parentheses, is the **Reynolds number** and is often written Re. It measures how turbulent the fluid flow is. To find the drag force F, we have to find the function f. It is too hard to determine fully – it would require solving the Navier–Stokes equations – but it might be possible in extreme cases. The extreme cases here are $Re \to 0$ and $Re \to \infty$.

Let's hope that the falling cones are in one of those limits! To decide, evaluate Re for the falling cone. From experience, even before you drop the cones to decide which falls faster, either cone falls at roughly $v \sim 1\,\mathrm{m\,s^{-1}}$. Its size is roughly $r \sim 0.1\,\mathrm{m}$. And the viscosity of the fluid (air) in which it falls is $\nu \sim 10^{-5}\,\mathrm{m^2\,s^{-1}}$, which you can find by looking it up in a table by an online search, or by applying these approximation methods to physics and engineering problems (the theme of another course and book on approximation). So

$$Re \sim \frac{\overbrace{0.1\,\mathrm{m}}^{r} \times \overbrace{1\,\mathrm{m\,s^{-1}}}^{v}}{\underbrace{10^{-5}\,\mathrm{m^2\,s^{-1}}}_{\nu}} \sim 10^4.$$

So $Re \gg 1$, and we are safe in looking just at that extreme case. Even if the estimate for the speed and size are inaccurate by, say, a factor of 3 each, the Reynolds number is at least 1000, still much larger than 1.

To decide what factors are important in the high-Reynolds-number limit, look at the form of the Reynolds number: rv/ν. One way to send it to infinity is the limit $\nu \to 0$. Viscosity, therefore, becomes irrelevant as $Re \to \infty$, and in that limit the drag force F should not depend on viscosity. Although the conclusion is mostly correct, there are subtle lies in the argument. To clarify these subtleties required two hundred years of mathematical and physical development in both theory and experiment. So I will skip the truth, and hope that you are content at least for the moment with almost-truth, especially since it gives the same answer as the truth.

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Let's look at how the requirement of independence from ν constrains the general dimensionless form:

$$\frac{F}{\rho v^2 r^2} = f(Re)$$

The left side does not contain viscosity ν. The right side might because Re contains ν. So if any Reynolds number shows up on the right side, then viscosity will appear on the right side, with no viscosity on the left side with which to cancel it. And that situation would violate the extreme-case result that, in the $Re \to \infty$ limit, the drag force is independent of viscosity. So the right side must be independent of Re. Since f depended only on the Reynolds number, which has just been stricken off the list of allowed dependencies, the right side $f(Re)$ is a dimensionless constant. Therefore,

$$\frac{F}{\rho v^2 r^2} = \text{dimensionless constant},$$

or

$$F \sim \rho v^2 r^2.$$

And now we have the result that we need to find the relative terminal velocity of the large and small cones. The cones reach terminal speed when the drag force balances the weight. The weight is proportional to the area of the paper, so it is proportional to r^2. The drag force is also proportional to r^2, as we just found. To summarize:

$$\underbrace{\rho v^2 r^2}_{F} \propto \underbrace{r^2}_{\text{weight}}.$$

The factor of r^2 on each side divides out, so

$$v^2 \propto \frac{1}{\rho},$$

showing that

> The cones' terminal velocity is independent of its size.

That result is indeed what we found in class by doing the experiment. So, without having to solve the Navier–Stokes differential equations, experiment and cheap theory agree!

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2.8 What you have learned

The main theme of this chapter is the recipe for extreme-cases reasoning for checking and guessing the answers to complicated problems:

1. Pick an extreme value where the result is easy to determine without solving the full problem; for example, for the ellipse, its area is easy when $a = 0$ or $b = 0$.

2. For that extreme case, determine the result. For the ellipse, the area is zero when either $a = 0$ or $b = 0$.

3. Determine the prediction in this extreme case, and compare it with the actual value from the second step. So, for the ellipse, any candidate for the area had better go to zero when $a = 0$ or $b = 0$.

Extreme cases also complements the technique of dimensions, once the problems become too complicated for the naive methods of the previous chapter. That symbiosis was illustrated in computing the relative terminal velocities of the falling cones. The general recipe is based on the maxim that **You can write any true statement about the world using dimensionless groups.** It leads to the following problem-solving plan for finding, say, drag force F:

1. Find the quantities on which F depends, and find the dimensions of F and of those quantities.

2. Make dimensionless groups from those quantities.

3. Write the result in general dimensionless form:

$$\text{group containing } F = f(\text{other groups}).$$

4. Use extreme-cases reasoning to guess the form of the dimensionless function f.

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