1 Dimensions

Dimensions, often called units, are familiar creatures in physics and engineering. They are also helpful in mathematics, as I hope to show you with examples from differentiation, integration, and differential equations.

1.1 Free fall

Dimensions are often neglected in mathematics. Calculus textbooks state many problems in this form:

> A ball falls from a height of h **feet**. Neglecting air resistance, estimate its speed when it hits the ground, given a gravitational acceleration of g **feet per second squared**.

The units, highlighted with boldface type, have been separated from g or h, making g and h pure numbers. That artificial purity ties one hand behind your back, and to find the speed you are almost forced to solve this differential equation:

$$\frac{d^2y}{dt^2} = -g, \text{with } y(0) = h \text{ and } \dot{y}(0) = 0,$$

where $y(t)$ is the ball's height at time t, $\dot{y}(t)$ is its velocity, and g is the strength of gravity (an acceleration). This second-order differential equation has the following solution, as you can check by differentiation:

$$\begin{aligned} \dot{y}(t) &= -gt, \\ y(t) &= -\frac{1}{2}gt^2 + h. \end{aligned}$$

The ball hits the ground when $y(t) = 0$, which happens when $t_0 = \sqrt{2h/g}$. The speed after that time is $\dot{y}(t) = -gt_0 = -\sqrt{2gh}$. This derivation has many spots to make algebra mistakes: for example, not taking the square root when solving for t_0, or dividing rather than multiplying by g when finding the speed.

Here's the same problem written so that dimensions help you:

> A ball falls from a height h. Neglecting air resistance, estimate its speed when it hits the ground, given a gravitational acceleration of g.

In this statement of the problem, the dimensions of h and g belong to the quantities themselves. The reunion helps you guess the final speed, without solving differential equations. The dimensions of h are now length or L for short. The dimensions of g are length per time squared or LT^{-2}; and the dimensions of speed are LT^{-1}. The only combination of g and h with the dimensions of speed is

$$\sqrt{gh} \times \text{dimensionless constant}.$$

An estimate for the speed is therefore

$$v \sim \sqrt{gh},$$

where the $\sim$ means 'equal except perhaps for a dimensionless constant'. Besides the minus sign (which you can guess) and the dimensionless factor $\sqrt{2}$, the dimensions method gives the same answer as does solving the differential equation – and more quickly, with fewer places to make algebra mistakes. The moral is:

> Do not rob a quantity of its intrinsic dimensions.

Its dimensions can guide you to correct answers or can help you check proposed answers.

1.2 Integration

If ignoring known dimensions, as in the first statement of the free-fall problem, hinders you in solving problems, the opposite policy – specifying unknown dimensions – can aid you in solving problems. You may know this Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

What is the value of

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx,$$

where α is a constant? The integration variable is x so after you evaluate the integral over the limits, the x disappears; but α remains. The result contains only α and maybe dimensionless numbers, so α is the only quantity in the result that could have dimensions. For dimensional analysis to have a prayer of helping, α needs dimensions. Otherwise you cannot say whether, for example, the result should contain α or contain α^2; both choices have identical dimensions. Guessing the answer happens in three steps: (1) specifying the dimensions of α, (2) finding the dimensions of the result, and (3) using α to make a quantity with the dimensions of the result.

In the first step, finding the dimensions of α, it is more intuitive to specify the dimensions of the integration variable x, and let that specification decide the dimensions of α. Pretend that x is a length, as its name suggests. Its dimensions and the exponent $-\alpha x^2$ together determine the dimensions of α. An exponent, such as the 7 in 2^7, says how many times to multiply a quantity by itself. The notion 'how many times' is a pure number; the number might be negative or fractional or both, but it is a pure number:

> An exponent must be dimensionless.

Therefore αx^2 is dimensionless, and the dimensions of α are L^{-2}. A convenient shorthand for those words is

$$[\alpha] = \mathrm{L}^{-2},$$

where [quantity] stands for the dimensions of the quantity.

The second step is to find the dimensions of the result. The left and right sides of an equality have the same dimensions, so the dimensions of the result are the dimensions of the integral itself:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx.$$

What are the dimensions of an integral? An integral sign is an elongated 'S', standing for *Summe*, the German word for sum. The main principle of dimensions is:

> You cannot add apples to oranges.

Two consequences are that every term in a sum has identical dimensions and that the dimensions of a sum are the dimensions of any term. Similarly, given the kinship of summation and integration, the dimensions of the integral are the dimensions of $e^{-\alpha x^2}\,dx$. The exponential, despite the fierce-looking exponent of $-\alpha x^2$, is just the pure number e multiplied by itself several times. Since e has no dimensions, e^{anything} has no dimensions. So the exponential factor contributes no dimensions to the integral. However, the dx might contribute dimensions. How do you know the dimensions of dx? If you read d as 'a little bit of', then dx becomes 'a little bit of x'. A little bit of length is still a length. More generally:

> dx has the same dimensions as x.

The product of the exponential and dx therefore has dimensions of length, as does the integral – because summation and its cousin, integration, cannot change dimensions.

The third step is to use α to construct a quantity with the dimensions of the result, which is a length. The only way to make a length is $\alpha^{-1/2}$, plus perhaps the usual dimensionless constant. So

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx \sim \frac{1}{\sqrt{\alpha}}.$$

The twiddle $\sim$ means 'equal except perhaps for a dimensionless constant'. The missing constant is determined by setting $\alpha = 1$ and reproducing the original integral:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Setting $\alpha = 1$ is a cheap trick. Several paragraphs preceding exhorted you not to ignore the dimensions of quantities; other paragraphs were devoted to deducing that α had dimensions of L^{-2}; and now we pretend that α, like 1, is dimensionless?! But the cheap trick is useful. It tells you that the missing dimensionless constant is $\sqrt{\pi}$, so

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}.$$

The moral of the preceding example is:

> Assign dimensions to quantities with unspecified dimensions.

In this example, by assigning dimensions to x and α, we got enough information to guess the integral.

1.3 Taylor and MacLaurin series

The preceding example applied dimensions to integrals. Dimensions also help you remember Taylor series, a result based on derivatives. The idea of Taylor series is that if you know a function and all its derivatives at one point, you can approximate the function at other points. As an example, take $f(x) = \sqrt{x}$. You can use Taylor series to approximate $\sqrt{10}$ by knowing $f(9)$ and all the derivatives $f'(9)$, $f''(9)$, ...

The MacLaurin series, a special case of Taylor series when you know $f(0)$, $f'(0)$, ..., looks like:

$$f(x) = f(0) + \text{stuff}$$

What is the missing stuff? The first principle of dimensions can help, that you cannot add apples to oranges, so all terms in a sum have identical dimensions. The first term is the zeroth derivative $f(0)$. The first term hidden in the 'stuff' involves the first derivative $f'(0)$, and this new term must have the same dimensions as $f(0)$. To draw a conclusion from this sameness requires understanding how differentiation affects dimensions.

In the more familiar notation using differentials,

$$f'(x) = \frac{df}{dx}.$$

So the derivative is a quotient of df and dx. You can never – well, with apologies to Gilbert & Sullivan, hardly ever – go astray if you read d as 'a little bit of'. So df means 'a little bit of f', dx means 'a little bit of x', and

$$f'(x) = \frac{df}{dx} = \frac{\text{a little bit of } f}{\text{a little bit of } x}.$$

Using the [quantity] notation to stand for the dimensions of the quantity, the dimensions of $f'(x)$ are:

$$[f'(x)] = \frac{[\text{a little bit of } f]}{[\text{a little bit of } x]}.$$

Since a little bit of a quantity has the same dimensions as the quantity itself,

$$[f'(x)] = \frac{[\text{a little bit of } f]}{[\text{a little bit of } x]} = \frac{[f]}{[x]}.$$

Differentiating with respect to x is, for the purposes of dimensional analysis, equivalent to dividing by x.

So $f'(x)$ has the same dimensions as f/x.

This strange conclusion is worth testing with a familiar example. Take distance x as the function to differentiate, and time as the independent variable. The derivative of $x(t)$ is $\dot{x}(t) = dx/dt$. [Where did the prime go, as in $x'(t)$? When the independent variable is time, a dot instead of a prime is used to indicate differentiation.] Are the dimensions of $\dot{x}(t)$ the same as the dimensions of x/t? The derivative $\dot{x}(t)$ is velocity, which has dimensions of length per time or LT^{-1}. The quotient x/t also has dimensions of length per time. So this example supports the highlighted conclusion.

The conclusion constrains the missing terms in the MacLaurin series. The first missing term involves $f'(0)$, and the term must have the same dimensions as $f(0)$. It doesn't matter what dimensions you give to $f(x)$; the principle of not adding apples to oranges applies whatever the dimensions of $f(x)$. Since its dimensions do not matter, choose a convenient one, that $f(x)$ is a volume. Do not, however, let x remain unclothed with dimensions. If you leave it bare, dimensions cannot help you guess the form of the MacLaurin series: If x is dimensionless, then differentiating with respect to x does not change the dimensions of the derivatives. Instead, pick convenient dimensions for x; it does not matter which dimensions, so long as x has some dimensions. Since the symbol x often represents a length, imagine that this x is also a length.

The first derivative $f'(0)$ has dimensions of volume over length, which is length squared. To match $f(0)$, the derivative needs one more power of length. The most natural object to provide the missing length is x itself. As a guess, the first-derivative term should be $xf'(0)$. It could also be $xf'(0)/2$, or $xf'(0)$ multiplied by any dimensionless constant. Dimensional analysis cannot tell you that number, but it turns out to be 1. The series so far is:

$$f(x) = f(0) + xf'(0) + \cdots.$$

Each successive term in a MacLaurin (or Taylor) series contains a successively higher derivative. The first term used $f(0)$, the zeroth derivative. The second term used $f'(0)$, the first derivative. The third term should use the second derivative $f''(0)$. The dimensions of the second derivative are volume over length squared. because each derivative divides f by one length. Compared to the volume, $f''(0)$ lacks two lengths. The most natural quantity to replace those lengths is x^2, so the term should be $x^2f''(0)$. It could be multiplied by a dimensionless constant, which this method cannot find. That number turns out to be $1/2$, and the term is $x^2f''(0)/2$. The series is now

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2f''(0) + \cdots.$$

You can guess the pattern. The next term uses $f^{(3)}(0)$, the third derivative. It is multiplied by x^3 to fix the dimensions and by a dimensionless constant that turns out to be $1/6$:

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2f''(0) + \frac{1}{6}x^3f^{(3)}(0) + \cdots$$

The general term is

$$\frac{x^nf^{(n)}(0)}{n!},$$

for reasons that will become clearer in ?? on analogies and operators. This example illustrates how, if you remember a few details about MacLaurin series – for example, that each term has successively higher derivatives – then dimensional analysis can fill in the remainder.

1.4 Cheap differentiation

The relation $[f'(x)] = [f] / [x]$ suggests a way to estimate the *size* of derivatives. Here is the differential equation that describes the oscillations of a mass connected to a spring:

$$m\frac{d^2x}{dt^2} + kx = 0,$$

where m is the mass, x is its position, t is time, and k is the spring constant. In the first term, the second derivative d^2x/dt^2 is the acceleration a of the mass, so $m(d^2x/dt^2)$ is ma or the force. And the second term, kx, is the force exerted by the spring. In working out what the terms mean, we have also checked that the terms have the same dimensions (here, dimensions of force). So the equation is at least dimensionally correct.

Here's how to estimate the size of each term. The dimensions of d^2x/dt^2 comes from dividing the dimensions of x by the dimensions of t^2. The size of d^2x/dt^2 is estimated by dividing the size of x by the size of t^2. Why not instead divide the dimensions of x^2 by those of t^2? The numerator, after all, has a d^2 in it. To answer that question, return to the maxim: d means 'a little bit of'. So dx means 'a little bit of x', and $d^2x = d(dx)$ means 'a little bit of a little bit of x'. The numerator, therefore does not have anything to do with x^2. Instead, it has the same dimensions as x. Another way of saying the same idea is that differentiation is a linear operation.

Even if x/t^2 is a rough estimate for the second derivative, x and t are changing: How do you know what x and t to use in the quotient? For x, which is in the numerator, use a **typical value** of x. A typical value is the oscillation amplitude x_0. For t, which is in the denominator, use the time in which the numerator changes significantly. That time – call it τ – is related to the oscillation period. So

$$\frac{dx}{dt} \sim \frac{\text{typical } x}{\tau} \sim \frac{x_0}{\tau},$$

and

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) \sim \frac{1}{\tau}\frac{x_0}{\tau} = \frac{x_0}{\tau^2}.$$

Now we can estimate both terms in the differential equation:

$$m\frac{d^2x}{dt^2} \sim m\frac{x_0}{\tau^2}.$$
$$kx \sim kx_0,$$

The differential equation says that the two terms add to zero, so their sizes are comparable:

$$m\frac{x_0}{\tau^2} \sim kx_0.$$

Both sides contain one power of the amplitude x_0, so it divides out. That cancellation always happens in a linear differential equation. With x_0 gone, it cannot affect the upcoming estimate for τ. So

> In ideal spring motion – so-called simple harmonic motion – the oscillation period is **independent** of amplitude.

After cancelling the x_0, the leftover is $k \sim m/\tau^2$ or $\tau \sim \sqrt{m/k}$. A quantity related to the time τ is its reciprocal $\omega = \tau^{-1}$, which has dimensions of inverse time or T^{-1}. Those dimensions are the dimensions of frequency. So

$$\omega = \tau^{-1} \sim \sqrt{\frac{k}{m}}.$$

When you solve the differential equation honestly, this ω is exactly the angular frequency (angle per time) of the oscillations. The missing constant, which dimensional analysis cannot compute, is 1. In this case, dimensional analysis, cheap though it may be, gives the exact frequency.

1.5 Free fall revisited

The ball that fell a height h was released from rest. What if it had an initial velocity v_0. What is its impact velocity v_{final}?

1.6 What you have learned

- Preserve dimensions in quantities with dimensions: Do not write 'g meters per second squared'; write g.
- Choose dimensions for quantities with arbitrary dimensions, like for x and α in
$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx.$$
- Exponents are dimensionless.
- You cannot add apples to oranges: Every term in an equation or sum has identical dimensions. Another consequence is that both sides of an equation have identical dimensions.
- The dimensions of an integral are the dimensions of everything inside it, including the dx. This principle helps you guess integrals such as the general Gaussian integral with $-\alpha x^2$ in the exponent.

- The dimensions of a derivative $f'(x)$ are the dimensions of f/x. This principle helps reconstruct formulas based on derivatives, such as Taylor or MacLaurin series.

- The size of df/dx is roughly

$$\frac{\text{typical size of } f}{x \text{ interval over which } f \text{ changes significantly}}$$

See the short and sweet book by Cipra [1] for further practice with dimensions and with rough-and-ready mathematics reasoning.