

Proof of Sarkovskii's Theorem and Its Converse

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1 Introduction

When determining whether a function is chaotic under iteration, it is necessary to prove that the set of all periodic points is a dense subset of the space on which the function acts. If the function is simple enough, it may be possible to explicitly construct a periodic orbit that comes within an arbitrary distance of any given point. However, often this is not possible, and so a more general approach is needed.

Sarkovskii's Theorem provides a means of proving the existence of infinitely many cycles, each with different period, provided that we can find a cycle of length $k \neq 2^n, n \in \mathbb{Z}^+$. In this paper, we will first prove Sarkovskii's Theorem and then go on to prove its converse.

2 Sarkovskii's Theorem

2.1 Sarkovskii's Ordering

Before formally stating Sarkovskii's Theorem, it is necessary to define Sarkovskii's Ordering. This ordering of the natural numbers begins with all odd numbers, written in increasing order. These are followed by 2 times the odds, 2^2 times the odds, 2^3 times the odds, and so on. The powers of 2 come last, in decreasing order. This ordering can be written as follows:

$$\begin{aligned} &3, 5, 7, 9, \dots \\ &2 \cdot 3, 2 \cdot 5, 2 \cdot 7, \dots \\ &2^2 \cdot 3, 2^2 \cdot 5, 2^2 \cdot 7, \dots \\ &2^3 \cdot 3, 2^3 \cdot 5, 2^3 \cdot 7, \dots \\ &\vdots \\ &\dots, 2^n, \dots, 2^3, 2^2, 2, 1 \end{aligned}$$

Definition 1. Given $x, y \in \mathbb{Z}^+$, if x precedes y in the Sarkovskii Ordering, we write $x \triangleright y$. If x follows y in the ordering, we write $x \triangleleft y$.

We could therefore write Sarkovskii's Ordering as

$$\begin{aligned} &3 \triangleright 5 \triangleright 7 \triangleright \dots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright \dots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright \dots \\ &\triangleright 2^3 \cdot 3 \triangleright 2^3 \cdot 5 \triangleright \dots \triangleright 2^n \dots \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1 \end{aligned}$$

2.2 Sarkovskii's Theorem

Theorem 1 (Sarkovskii's Theorem). *Suppose $f : \mathbb{R} \leftarrow \mathbb{R}$ is continuous and has a periodic point of prime period n . If $n \triangleright k$, then f also has a periodic point of prime period k .*

Proof

To prove Sarkovskii's Theorem, we first need to establish a few basic lemmas. To prove Sarkovskii's Theorem, we will divide the proof into 7 cases.

Case 2.1. *If f has a point with period n , where n is odd, then f has a point with period $k > n$, k odd.*

Case 2.2. *If f has a point with period n , n odd, then f has a point with period k , k even.*

Case 2.3. *If f has a point with period n , n even, then f has a point with period 2.*

For the proofs of these three cases, see Robert Devaney's "An Introduction to Chaotic Dynamical Systems," pp. 63-65.

Case 2.4. *If f has a point with period $n = 2^m$, then f has a point with period $k = 2^l$, $l < m$.*

Proof. Let $k = 2^l$, $l < m$. Let $g = f^{k/2}$. By assumption, g has a point with period 2^{m-l+1} . Therefore, g has a point with period 2, by Case 3. This point has period $k = 2^l$ under f . \square

Case 2.5. *If f has a point with period $n = p \cdot 2^m$, p odd and $m \geq 0$, then f has a point with period $k = q \cdot 2^m$, $q > p$ and q odd.*

Proof. Let $g = f^{2^m}$. Since f has a point with period $p \cdot 2^m$, that point has period p under g . So by Case 1, there is a point with period q under g , for all q odd and $q > p$. This point has period $q \cdot 2^m$ under f . \square

Case 2.6. *If f has a point with period $n = p \cdot 2^m$, p odd and $m \geq 0$, then f has a point with period $k = q \cdot 2^l$, $l > m$.*

Proof. Let $g = f^m$. By assumption, g has a point with period p . So by Case 2, g has a point with period $q \cdot 2^{l-m}$, $l > m$. This point has period $q \cdot 2^l$ under f . \square

Case 2.7. *If f has a point with period $n = p \cdot 2^m$, p odd and $m \geq 0$, then f has a point with period $k = 2^l$, $l > m$.*

Proof. By Case 6, f has a point with period $2^{l-m} \cdot 2^m = 2^l$. \square

We have thus proven Sarkovskii's Theorem.

3 The Converse of Sarkovskii's Theorem

Theorem 2. *Given $n \in \mathbb{Z}^+$, there exists a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that given any $k \triangleleft n$, f does not have a point with period k .*

3.1 Proof

Given $n \in \mathbb{Z}^+$, it suffices to show that there exists a function f that does not have period k , where k is the number immediately preceding n in Sarkovskii's Ordering.

We will again divide the proof into multiple cases, in each instance constructing a continuous function which does not have the period immediately preceding n in the ordering.

Case 3.1. Given $n \in \mathbb{Z}^+$, $\exists f : \mathbb{R} \rightarrow \mathbb{R}$ such that f is continuous and has a point with period $2n + 1$ but no point of period $2n - 1$.

Proof. We will first construct a continuous function $f : [1, 2n + 1] \rightarrow [1, 2n + 1]$ and then extend the domain to include the entire set of reals.

Let $f : [1, 2n + 1] \rightarrow [1, 2n + 1]$ be defined by

$$\begin{aligned} f(1) = n + 1 & & f(2) = 2n + 1 & & f(3) = 2n & & f(4) = 2n - 1 & \dots & f(n) = n + 3 \\ f(n + 1) = n + 2 & & f(n + 2) = n & & f(n + 3) = n - 1 & \dots & f(2n + 1) = 1 \end{aligned}$$

All of the integers in $[1, 2n + 1]$ are then of period $2n + 1$, by construction. Let f be defined linearly within any interval $[j, j + 1]$.

To show that there are no points with period $2n - 1$, we will first show that such a point cannot be in $[1, 2]$. We must therefore iterate $[1, 2]$ $2n - 1$ times.

$$\begin{aligned} [1, 2] &\xrightarrow{f} [n + 1, 2n + 1] \xrightarrow{f} [1, n + 2] \\ &\xrightarrow{f} [n, 2n + 1] \xrightarrow{f} [1, n + 3] \\ &\xrightarrow{f} [n - 1, 2n + 1] \xrightarrow{f} \dots \xrightarrow{f} [1, 2n] \\ &\xrightarrow{f} [2, 2n + 1] \end{aligned}$$

Since $f^{2n-1}([1, 2]) \cap [1, 2] = 2$, and 2 has period $2n + 1$, $[1, 2]$ contains no points of period $2n - 1$.

To show that none of the other intervals contain a point of period $2n - 1$, we will show that all but one other interval is eventually mapped into $[1, 2]$. The last interval will be dealt with separately.

Given any interval $[j, j + 1]$ with $j < n + 1$, $f([j, j + 1]) = [k, k + 1]$ for some $k > n + 2$.

Given any interval $[j, j + 1]$ with $j > n + 2$, $f^2([j, j + 1]) = [j + 1, j + 2]$.

So there is an iterate f that maps $[j, j + 1]$ into $[1, 2]$, for all $j \neq n + 1$, since eventually there is an iterate that maps the interval to $[2n, 2n + 1]$ and the next iterate will be $[1, 2]$.

For the interval $[n + 1, n + 2]$, first note that $f([n + 1, n + 2]) = [n, n + 2]$. Hence, we have two cases. The first case is that $f^k(x) \in [n, n + 1]$ for some $k \in \mathbb{Z}^+$. Then, as shown above, some iterate of x lies in $[1, 2]$, so x cannot have period $2n - 1$.

The second case is that the orbit of x is contained entirely in $[n + 1, n + 2]$. In this interval, $|f'| = 2 > 1$, so x must be a fixed point.

To extend the domain of f to include the entire set of reals, define f' as follows:

$$f'(x) = \begin{cases} f(x), & 1 < x < 2n + 1 \\ x, & x < 1 \text{ or } x > 2n + 1 \end{cases}$$

All points outside the domain of f are fixed points, so all cycles of period greater than 1 must be in the interval $[1, 2n + 1]$. f' is thus the desired extension of f . \square

Case 3.2. Given $n, k \in \mathbb{Z}^+$, $\exists f : \mathbb{R} \rightarrow \mathbb{R}$ such that f is continuous and has a point with period $2^k(2n + 1)$ but no point of period $2^k(2n - 1)$.

Case 3.3. Given $k \in \mathbb{Z}^+$, $\exists f : \mathbb{R} \rightarrow \mathbb{R}$ such that f has a point with period 2^k , but no point with period 2^{k+1} .

Case 3.4. Given $k \in \mathbb{Z}^+$, $\exists f : \mathbb{R} \rightarrow \mathbb{R}$ such that f has a point with period $2^k \cdot 3$, but no point with period $2^{k-1} \cdot n$, where n is any odd number.

For the proofs of these three cases, see Saber Elaydi's "On a Converse of Sharkovsky's Theorem."
This completes the proof of the converse to Sarkovskii's Theorem. \square