

Sarkovskii's Theorem and its Converse

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1 Introduction

Sarkovskii's Theorem is an important result about the kinds of periodic points that a continuous function on a closed interval can have. First we define the Sarkovskii ordering of the natural numbers to be the following, where $a \triangleright b$ means a appears before b in this ordering:

$$\begin{aligned} &3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright 11 \triangleright \dots \\ &2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright \dots \\ &2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright \dots \\ &\dots 2^4 \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1. \end{aligned}$$

The first row consists of the ordered odd numbers, then each row consists of the the odd integers multiplied by increasing multiples of powers of two, and lastly we list the powers of two in decreasing order finishing with 1. Sarkovskii's Theorem says that if f is a continuous function from the interval I onto itself, then if $n \triangleright r$ and f has a point of period n , f has a point of period r .

Professor Devaney offers a partial proof of Sarkovskii's Theorem [1, 60-68]. We present the last case of the proof, the case $n = 2^m p$, using Devaney's results, and we additionally present a proof of the Converse of Sarkovskii's Theorem, which states that for every n there is a function with a cycle of period n and no cycles with periods higher than n in Sarkovskii's ordering of the natural numbers. Our proof of the Converse to Sarkovskii's theorem is largely based on Elaydi's proof [2].

2 Sarkovskii's Theorem

2.1 Case $n = 2^m p$

We conclude the proof of Sarkovskii's Theorem starting from two previously proven cases, $n = p$ for p an odd number bigger than 1 and $n = 2^m$. The Facts we have are that if a function f has a point of period n , then it has points of the following periods:

- I. $n = p$ implies all periods q odd such that $q > p$
- II. $n = p$ implies all even periods
- III. $n = 2^m$ implies all periods 2^l for any l such that $0 \leq l < m$

Proposition. If a function f has a point of period $n = 2^m p$ where p is odd bigger than 1 and m is any positive integer, then f has points with the following periods:

- (1) $2^m q$ for q odd such that $q > p$
- (2) $2^l q$ for l such that $l > m$ and any q odd, $q \neq 1$
- (3) 2^l for any $l \geq 0$

Proof: (1) Let f have a point of prime period $n = 2^m p$, and consider the function $g = f^{2^m}$. Then g has a point of period p , which by Fact I implies g has points of all periods q odd such that $q > p$, so that f has points of periods $2^m q$ for all q odd such that $q > p$.

(2) Consider f and g as in the last case. Fact II implies g has points of all even periods, including periods of the form $2^k q$ for any integer $k > 0$ and any q odd, $q \neq 1$. Then f has points of all periods $2^l q$ where $l = m + k$, so l is any integer such that $l > m$ and q is any odd number not equal to 1.

(3) With f and g as before, Fact II implies g has points with all periods of the form 2^k for any integer $k \geq 0$. Hence, f has points with periods 2^l , where $l = k + m$, so $l \geq m$. Fact III implies that f also has all lower powers of 2, so f has points with periods 2^l for any $l \geq 0$. This completes the proof of the theorem.

3 The Converse of Sarkovskii's Theorem

We will divide the proof in four cases, when $n = 2k + 1$, $n = 2^k p$, $n = 2^k$, and $n = 3 \cdot 2^k$ where k is an arbitrary positive integer and p an arbitrary odd number.

3.1 Case $n = 2k + 1$.

We construct a continuous function f that maps $[1, n]$ to itself that has a cycle of period $2k + 1$ but not of $2k - 1$. Let

$$f(x) = \left\{ \begin{array}{ll} kx + 1 & \text{if } 1 \leq x \leq 2, \\ 2k + 3 - x & \text{if } 2 < x \leq k + 1, \\ 3k + 4 - 2x & \text{if } k + 1 < x \leq k + 2, \\ 2k + 2 - x & \text{if } k + 2 < x \leq 2k + 1. \end{array} \right\}.$$

One can verify the endpoints of the corresponding lines above meet. Thus f is a continuous function. The cycle of period $2k + 1$ is

$$1 \rightarrow k + 1 \rightarrow k + 2 \rightarrow k \rightarrow k + 3 \rightarrow k - 1 \rightarrow \dots \rightarrow 2k + 1 \rightarrow 1$$

where

$$k + 1 + i \rightarrow k + 1 - i \rightarrow k + 1 + (i - 1) \dots$$

holds for the integers i such that $1 \leq i \leq k$.

To compute $f^{2k-1}[1, 2]$, we note that f maps the following closed intervals in the following way:

$$[1, 2] \rightarrow [k + 1, 2k + 1] \rightarrow [1, k + 2] \rightarrow [k, 2k + 1] \rightarrow \dots \rightarrow [1, 2k] \rightarrow [2, 2k + 1]$$

where

$$[1, k + i] \rightarrow [k + 2 - i, 2k + 1] \rightarrow [1, k + 1 + i]$$

holds for $2 \leq i \leq k$. It follows that

$$f^{2k-1}[1, 2] = [2, 2k + 1].$$

The point $x = 2$, the intersection of the last two intervals, is a part of the $(2k + 1)$ -cycle, so it is not a fixed point of f^{2k-1} , and this function has no other fixed points in $[1, 2]$.

We can similarly verify that none of the other intervals of $[1, 2k + 1]$ of the form $[i, i + 1]$ has a fixed point of f^{2k-1} except for $[k + 1, k + 2]$ because f performs the following maps:

$$[k + 1, k + 2] \rightarrow [k, k + 2] \rightarrow [k, k + 3] \rightarrow [k - 1, k + 3] \rightarrow \dots \rightarrow [1, 2k + 1]$$

where

$$[k + 1 - i, k + 1 + i] \rightarrow [k + 1 - i, k + 1 + (i + 1)] \rightarrow [k + 1 - (i + 1), k + 1 + (i + 1)]$$

holds for $1 \leq i \leq k - 1$. It follows that $f^{2k-1}[k + 1, k + 2] = [1, 2k + 1]$ so f^{2k-1} must have a fixed point in that interval. We will show that point is unique and hence it is the fixed point of f , which does not belong to any cycle. Note that f is monotonically decreasing over each of the intervals in the sequence of mappings above. It follows, since the composition of an odd number of monotonically decreasing functions is monotonically decreasing, that the fixed point is indeed unique, since f^{2k-1} crosses the line $y = x$ only once as it goes through the interval $[k + 1, k + 2]$. Hence f has no cycles of period $2k - 1$.

3.2 Case $n = 2^k p$.

We construct a continuous function g on a closed interval that has a cycle of period $2^k p$ but not of $2^k(p - 2)$. In this and the following constructions we use the "trick" double map defined by Devaney that generates a function g from any continuous function f such that g has cycles of exactly double the periods of all cycles of f and adds a fixed point without adding any other cycles [1, 67].

Let f be a continuous function with a cycle of period p odd but no cycle of period $p - 2$, which exists by the last case. Let g be the function obtained by applying the double map k times to f . Then g has exactly a cycle of period $n = 2^k p$ but not of $n = 2^k(p - 2)$, because otherwise f would have had a cycle of period $p - 2$. Hence, g is the desired function.

3.3 Case $n = 2^k$.

In this case we want a continuous function on a closed interval that has points of period $n = 2^k$ but none of period $n = 2^{k+1}$. We can generate such functions using the double map.

Let $f(x) = 3 - x$ map $[1, 2]$ to itself. All of the points in that interval have prime period 2 except for the fixed point $3/2$. Applying the double map $k - 1$ times to f gives us the desired function g because it generates a function with a point of period $n = 2^k$ but none of $n = 2^{k+1}$, for otherwise f would have had a point of prime period 4.

3.4 Case $n = 3 \cdot 2^k$.

This last case is the only one in which n has no immediate predecessor. We want to generate a function that has points with prime period $n = 3 \cdot 2^k$ but none of the form $n = 2^{k-1}p$ for any positive odd number p . These functions can be inductively defined using the double map.

Let $f(x)$ be defined from $[1, 3]$ to itself by $f(1) = 2$, $f(2) = 3$ and $f(3) = 1$ connected by lines on $[1, 2]$ and $[2, 3]$. It clearly has a point of prime period 3, so this function satisfies the case $n = 3$ ($k = 0$). Applying the double map k times to f gives us the desired function g because it has a point of period $n = 3 \cdot 2^k$ but none of the form $n = p \cdot 2^{k-1}$, since there are no integers q such that $2^k q = 2^{k-1} p$.

This completes the proof of the converse to Sarkovskii's Theorem.

References

- [1] Devaney, Robert L. *An Introduction to Chaotic Dynamical Systems*, Second Edition. Addison-Wesley, Redwood City, California, 1989.
- [2] Elaydi, Saber. *On a Converse of Sharkovsky's Theorem*. American Math. Monthly 103 (May 1996), 386-392.