

Lab 2: Rates of Convergence

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1 Introduction

The goal of this lab was to study how the magnitude of the derivative of a function at a fixed point, $|F'(p)|$, affected the rate at which the function converged to a fixed point under iteration.

To study this characteristic, I tested twelve functions, all of which were known to converge, using a computerized function interater (<http://math.bu.edu/DYSYS/applets/Iteration.html>). For each, I determined where the fixed point was, what the derivative was at the fixed point, and if necessary, the second derivative at the fixed point. Lastly, I recorded how many iterations were necessary to “reach” the fixed point starting at $x_0 = 0.2$. “Reaching” the fixed point was defined to be coming within .001 of the actual fixed point, since the function never actually reaches the fixed point.

2 Analysis

Since all the functions that I tested were attracting, then by the proof of the Repelling Fixed Point Theorem, $|F'(p)| \leq 1$. I can therefore divide our test cases into two groups, those with derivative less than one, and those with derivative equal to one.

2.1 Attracting Fixed Points: $|F'(p)| < 1$

In this group, I have seven functions. These functions, and the calculations for each are listed in Table 2.

The unifying feature of these functions is that the function always goes to the fixed point very quickly. In every case, the function was within the desired interval in under 10 iterations.

2.2 Neutral, Weakly Attracting Fixed Points: $|F''(p)| = 1$

The remaining five functions that I tested all possessed weakly attractive fixed points. The numerical calculations are listed in Table 3.

c	$ F'(p) $	Iterations
2.9	.90	60
2.99	.99	341
2.999	.999	1034

Table 1: Effect of varying c for $F(x) = cx(1 - x)$

With these functions, the orbit converged very slowly. In general, these functions required a number of iterations that was several orders of magnitude above the number needed for the attractive fixed points.

For the group of functions $cx(1-x)$, I carried out further analysis. To see how the behavior of F changes as $|F'(p)| \rightarrow 1$, I analyzed the family $cx(1-x)$ with $c \rightarrow 3$. As $c \rightarrow 3$, $|F'(p)| \rightarrow 1$ and the number of iterations necessary to get within .001 of the fixed point grows rapidly (Table 1).

Due to the number of properties that are changed whenever you vary the parameter of these functions, I was not able to discern a pattern relating rate of convergence to higher-order derivatives.

Table 2: Attracting Fixed Points: $|F'(p)| < 1$

$F(x)$	Fixed Point (p)	$ F'(p) $	Iterations
x^2	0	0	3
$0.4x(1-x)$	0	0.4	6
$1.6x(1-x)$	0.375	0.4	8
$2x(1-x)$.5	0	4
$2.4x(1-x)$.583	.4	6
$.4 \sin x$	0	.4	6

Table 3: Weakly Attracting Fixed Points: $|F'(p)| = 1$

$F(x)$	Fixed Point (p)	$ F'(p) $	$ F''(p) $	Iterations
$x^2 + .25$.5	1	2	1000
$x^2 - .75$	-.5	1	2	>10000
$x(1-x)$	0	1	2	990
$3x(1-x)$	2/3	1	6	>10000
$\sin x$	0	1	0	>10000

Data