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PROFESSOR: Vasily. Vasily Strela who works now for Morgan Stanley, did his PhD here in the math department, and kindly said he would tell us about financial mathematics. So, it's all yours.

GUEST SPEAKER: Let me thank Professor Strang for giving this opportunity to talk here, and it feels very good to be back, be back to 18.086.

So, a few more words about myself. I've been Professor Strang's student in mathematics about ten years ago. So after receiving my PhD, I taught mathematics for a few years. Then ended up working for a financial institution for investment bank, Morgan Stanley in particular. I'm part of an analytic modeling group in fixed income division. What we are doing, we are doing multiplications in finance and modeling derivatives, fixed income derivatives. That's actually what I'm going to talk about today. I want to show how 18.086, it's a wonderful class which I admire a lot, which applications it has in the real world, and in particular in finance and derivatives pricing.

Let's start with a simple example, which actually comes not from finance, but rather from gambling. Well, let's look at horse racing or cockroach racing, if you prefer. Suppose there are two horses, and sure enough people bet on them and bookie uses a clever one, very scientific-minded guy and he made a very good research of previous history of these two horses. He found out that the first horse has 20% chance to win. The second horse has 80% chance to win. He is actually right about his knowledge about chances to win.

When that becomes general public, people who bet, they don't have access to all information, and the bets are split slightly differently. So the 10,000 is placed on the first horse, and 50,000 is placed on the second horse. Bookie, sticking to his

scientific knowledge, places the odds 4:1. Meaning that if the first horse wins then whoever put on the horse gets his money back and four times his money back on top of it. Or if the second horse wins then whoever put money on this horse will get the money back and 1/4 on top of it.

So, let's see. What are chances for bookie to win or lose in this situation? Well, if the first horse wins then he has to give back 10,000 plus 40,000, 50,000, and he got 60,000, so he gains 10,000. Well, good, good for him. On the other hand, if the second horse wins, then he has to give back 50,000 plus 1/4 of 50, which is 12,500, so 62.50 altogether, and he loses \$2,500. After many runs, the expected win or loss of the bookmaker is the probability of the first horse to win times the expected win, plus the probability of second horse to win times the expected loss, which turns out to be exactly zero.

So in each particular run, bookie may lose or win, but in the long run he expects to break even. On the other hand, if he would put the chances, he would set the odds according to the money bet, 5:1, what would be the outcome? Well, if the first horse wins he gives back 10,000 plus 50,000, 60,000 exactly the amount he collected. Or if the second horse wins, again, he gives back 50 plus 1/5 of that 60, he breaks even.

So no matter which horse wins in this scenario, the bookie breaks even. How bookie operates, well he actually charges a fee for each bet, right. So the second situation is much more preferable for him. When he doesn't care which horse wins, he just collects the fee. Well here, he may lose or gain money. This is quite beautiful observation, which we will see how it works in derivatives.

So now back to finance, back to derivatives. So we are actually interested in pricing a few financial derivatives, and what is a financial derivative? Well, a financial derivative is a contract, pay of which at maturity at some time c , depends on underlying security -- in our case, we always we will be talking about a stock as underlying security, and probably interest rates. What are the examples of financial derivatives? Well, the most simple example is where it's probably a forward

contract. Forward contracts is a contract when you agree to purchase the security for a price which is set today -- you've agreed to purchase the security in the future for the price agreed today.

Well, for example, if you needed 1,000 barrel of oil to keep your house, but not today but rather for the next winter, on that account you don't want to take the risks of waiting until the next winter and buying oil then, you would rather agree on the price now and pay it in the future and get the oil. What the price should be? What is the fair price for this contract? Well, we will see how to price it. Well, the few observation here is that this line represents the payout -- it's always useful to represent the payout graphically. This is just a straight line because the payout of our contract is f minus k at time c . This actually gives the current price of the contract for all different values of the underlying. Usually, the price of the forward contract is set such that for the current value of the underlying, the price of the contract is zero. It costs nothing to enter a forward contract, so that's why it intersects zero here.

What are other common derivatives? Another common derivative are call, and I put European call input here. Don't be confused by European or American. It has nothing to do with Europe or America, it has to do with the structure of the contract. European basically means that the contract expires at certain time c . American means that's you can exercise this contract at any time between now and future. We'll be talking only about European contracts.

So European call option is the contract which gives you the right, but not obligation, to purchase the underlying security at set price, k , which is called strike price, at a future, time c , which is expiration time. So if your security at time c ends up below k , below the strike, then sure enough there is no point of buying the security for a more expensive price. So the contract expires worthless. On the other hand, if your stock ends up being greater than k at expiration time c , then you would make money but my purchasing this stock for k dollars and your payout will be f minus k , and this is a graph of your payout. This line here, as we will see, is the current price of the contract, and we'll see how to obtain this line in a few minutes.

Another common contract is a put. While call was basically a bet that your stock will grow, right, the put is the bet that your stock will not grow. So, in this case, the put is the right, but not the obligation to sell the stock for a certain price k . Here is the payout, which is similar to the put, but just flipped. This is the current price of a put option. Calls and puts, being very common contracts are traded on exchanges -- Chicago exchange is probably the most common place for the calls and puts on stocks to trade. I just printed out a Bloomberg screen, which gives the information about a calls and puts on IBM stock. So I did it on March 8, and the IBM stock was trading at this time at \$81.14, and here are descriptions of the contract, they expire on 22nd of April, so it's pretty short-dated contract. They can go as far as two years from now, usually. Here is a set of strikes, and here are a set of prices. As you can see, there is no single price, there is always a bid and ask, and that's how dealers and brokers make their money -- like a bookie, they basically charge you a fee for selling or buying the contract. That's how the money are made -- they are made on this spread, but not on the price of the contract itself, because as we will see in a second, we actually can price the contract exactly, and there is no uncertainty once the price of the stock is set.

There are plenty of other options. Slightly more exotics contracts, either digital which pays either zero or one depending on where your stock ends up. It probably is not exchange-rated, also -- I'm not sure. There are hundreds, if not thousands, of exotic options where you can say that well, how much would be the right to purchase a stock for the maximum price between today and two years from now. So it will be past-dependent, depending on how the stock will go, the payout will be defined by this path. There are American options where you can decide your option any time between now and maturity, and so on and so forth.

So, just before we go into mathematics of pricing, just a few observations and statements. First of all, it turns out that thanks to developed mathematics, mathematical theory, if you make certain assumptions on the dynamics of the stock, then there is no uncertainty in the price of the option. You can say exactly how much the option costs now, and that's what provides, and this is a big driver for the

market. So dealers quote these contracts and there is a great agreement on the prices. The price of the derivative contract is defined completely by the stock price and not by risk preferences of the market participant. So it doesn't matter what are your views on the growth prospects of the stock. It will not effect the price of the derivative contract. As I said, so the mathematical part of it comes into giving the exact price without any uncertainty.

So let's consider a simple example now. Let's assume that we are in a very simple world. Well, first of all, in our world there are only three options -- the stock itself, the riskless with money market account, meaning that it is an account where we can either borrow money or invest money at the request rate r , and finally our derivative contract. Here we are not making any assumptions of what kind of derivative contract it is -- it could be forward, it could be call, it could be put, it can be anything.

Moreover, our world is so simple, but first of all, it's discrete, and second of all, there is only one time step to the expiration of power of contract, dt . Not only there is only one step left, we actually know exactly what our transition probabilities. There are only two states at the end, and we know the transition probability. So with probability p , we move from the state zero to the state one, and with probability of one minus p , we move to the state two. And I just noticed, because this is riskless money market account, it's the same in both cases. You just invest money and it grows with risk-free interest rate.

So, what can I say it about the price of our derivative f ? Well it's simple-minded -- well, let's start with the forward contract. We know what the payout in delta t of our forward contract will be, it will be just the difference between the stock price and our strike. Well, a simple-minded approach would be -- well, we know the transition probabilities, let's just compute the expected value of our contract, and that's what we would expect to get if there were many such experiments. Well, you take the probability of going to state one, you multiply by the payoff at stage one, take minus p for probability of going to state two. Multiply by the payout in that state two.

Sum them up and you get the expression, as I said, the common thing to choose

the strike is that the contract has zero value now, so you get your strike. Well, in particular you could say that if you research the market well and you know that the stock has equal probability of going up and down, then actually you expect your strike to be an average of end values of the stock. But as we can imagine, following our bookie example, this is not the right price. There is actually a definite price which doesn't depend on condition probability. Here is the reason why there is a definite price.

Well let's just consider a very simple strategy. Let's borrow just enough to purchase a stock. So let's borrow f zero dollars right now and buy the stock for this money. And let's enter the forward contract. Well, by definition forward contract has price zero now, so we enter the forward contract. Now, at the time dt when our contract expires, what happens? Well, we deliver our stock, which we already have in our hand for an exchange of k dollars. That's our forward contract. On the other hand, we have to repay our loan, and because it was a loan, it grew. It grew to s zero times e to the rdt .

Now let's see. What would happen if k was greater than f times e to the rdt ? Then we know for sure, we know now for sure, that we would make money. There is no uncertainty about it now. Similarly, if k is less than this value, then we know that we will lose money. That's not how the rational market works. If everybody knew that by setting this price you would make money, people would do it all day long and make infinite money. So there will be no other side of the market. So the price had to go down. So the only choice for k , the only market-implied choice, is that k has to be equal to f times e to the rdt . As you can see, it doesn't depend on transition probabilities at all. That's what market implies us. That's the price of forward contract, and that actually explains why when I was watching the forward contract, current price was just the straight line, it's just discounted payoff. The payout is linear, so just the parallel to the payoff.

That's the idea basically. The idea is to try to find such a portfolio of stock and the money market account with such a payout, which will exactly replicate the payoff of our derivative. If we found such of a portfolio, then we know for sure that the value

of this portfolio, the replicating portfolio today is equal to the value of the derivative, because otherwise, you would make or lose money riskless. That's no-arbitrage condition.

So, can we apply it to our general one step world? Well, if we have a general payout f , what we want to do, we want to form a replicating portfolio such that at expiration time, it will replicate our payouts. So we want to choose such constants a and b that such that the combination of stock and money market account in both states will replicate the payout of our option. Then if we are able to find such constants a and b , then we just look at the current price of the contract and it has to be equal to the current price of our derivative.

Well, but now a particular case, this is easy. It's just two linear equations with two unknowns, easily solved, and here is current price of our derivative. No matter what payout is -- I mean you just substitute the payout here, and if you know s_1 and s_2 , and s_2 that's it. A useful way to look at this, just to re-write this equation, is in this form, and then notice that actually the current price of our derivative can be viewed as a discounted expected payout of the derivative, but with very certain probability.

This probability, it doesn't come from statistical properties of the stock or from any research, it actually is defined by the market. So it's called a risk-neutral probability. So this probability doesn't depend on the views on the market by the market participants. An interesting observation is that actually the value of this stock, the discounted value of the stock is actually also is expected value of our outcomes on this risk-neutral probability. That's basically general idea.

Now let's move one notch up and try to apply these idea to continuous case. Well, if you live in continuous world now, we need to make some assumptions on the behavior of the stock. The very common assumption is that the dynamics of the stock is lognormal. Lognormal meaning that the logarithm of the stock is actually normally distributed. So, here u is some drift, σ is the volatility of our stock, and dw is the [INAUDIBLE] process, w 's the [INAUDIBLE] of process such that dw is normally distributed with mean zero and variance square root dt . Our approach

would be to find the replicating portfolio. What that mean, it means that we want to find such constants over time dt . So we assume that a and b are constant over the next step, dt , such that the change in our derivative is a linear combination with this constant of the change of our underlying security and the change of money market account.

Now we just need to look more closely at this equation. First of all, let's concentrate on df . So, f , our derivative is a function of stock value you time. But unfortunately, our stock value is stochastic, so the f is not that simple, and to write df we have to use a famous -- it's a formula from stochastic calculus, which actually is analagous of Taylor's formula for stochastic variables. Let's see. If our f will not be stochasticated, it would be completely deterministic and depend only on dt , then there would be no term and differential f is just the standard expression.

On the other hand, if we have dependence on stochastic variables, then we have to have more terms, and why this happens. Well, in very rough words is that because the order of magnitude of dw is higher than dt 's -- its square root of dt . So we have to make into account more terms, and particularly we have to -- to take in part next order of df squared. Formally, df square can be written this way, and again, very rough explanation is as follows. If we would square this equation there will be three terms there. One would come from the square of this term, and this would be of the order of dt squared, next order of magnitude -- much smaller than dt . The second term will be cross-product of dw dt . What order of magnitude that we are talking about, this is dt to the power $3/2$, again, much smaller than dt . On the other hand, the third term will be the square of dw , this is the order of magnitude of dt , so that's what dw is. And that's what either formula is about.

Now we are basically, now all terms here, and let me stress out that this term, dv it is not stochastic, it's completely deterministic because we know that d grows with the rate r , that's what it is. So we substitute all those terms into our replicating equation. We collect the terms. We get this equation. And again, there is the deterministic part, there is stochastic part. So the only way for this equation to hold is this term to be equal to this term, and this term to be equal to this term, and that's

what's written out here. So again, two equations, these two unknowns, and here is answer.

Finally, let's take a look at another part. Notice that this part of our equation is completely deterministic. So we know how it will grow. So basically, d of f minus σ^2 , which is d times dv , is r times d times dc . We know all other terms, we substitute them here, take something to the left hand side and we have this equation. So this is partial differential equation for our derivative f , as a function of f and t , of second order, and this equation is the famous Black-Scholes equation. It was derived by Fisher Black and Myron Scholes in their famous paper published in 1973. Myron Scholes and Robert Merton actually received Nobel Prize for deriving and solving this equation in '97. Black was already dead by the time. This is really the cornerstone of math finance. The cornerstone is because using the replicating portfolio, using this reasoning, we were able to find an exact equation for our derivative. So a few would remarks on Black-Scholes.

So first of all, we make some assumptions on the dynamic of the stock, but we never made any assumptions on our derivative. Which means that any derivative has to satisfy this equation, and that's very strong result. So if you assume that our stock was normal, which is not a bad assumption and agrees quite well with the market, then we basically in principle can price any derivative. We know the equation for any derivative. The other thing is that our Black-Scholes equation doesn't depend on the actual [UNINTELLIGIBLE] [μ] in the dynamics of our stock.

So again, it is the manifest of risk-neutral dynamic. Not only we wrote down the equation for our derivative, we also found a replicating portfolio. So in other words, we found a hedge and strategy, meaning that at any given time we can form this portfolio with rates a and b . If we hold both the derivative and both replicating portfolio all together, this is zero sum gain. We know that no matter where stock moves, we will not move money or gain money. So if we just charge bid offer on the derivative, if we charge a few on the contract, we can hedge ourself perfectly, buy the contract or sell the contract, hedge perfectly ourself and just make money on

the fee, that's it.

Finally, more mathematical remark is that actually after a few manipulations, a few change of variables, the Black-Scholes equation comes out to be just a heat question, which you already saw in this class. This is very good news. Why is this good news? Well, because heat equation is very well studied. So the solutions are well-known, and numerical methods, the ways to solve it, and particular the numerical ways to solve it are well-known. So we are in business.

But as any partial differential equation, the equation itself doesn't make much sense because to find a particular solution we need boundary and initial condition. And although any derivative which defines Black-Scholes equation, the final and boundary condition will vary from contract to contract. Here are a few examples of the final and boundary conditions. Here, an interesting remark that usually we would talk about initial condition, here we are talking about final condition if time goes in reverse -- we know the state of the world at the end, at expiration, not today.

So, here are final and boundary conditions for call and put, and let's look a little bit at the pictures for our call and put to see where they come from. So for example, for calls, well, this is our final condition, right, this is defined by the payout. Under the boundary condition, well, what happens, we put them at zero at an infinity, which shows [INAUDIBLE PHRASE], and why, well, because if stock hits zero then it stays in zero. That's what our dynamics show. So the value of power of contract at maturity will become zero. [UNINTELLIGIBLE PHRASE] the stock grows, grows to infinity, a good assumption to make is that actually it becomes similar to stock itself, so it would just become parallel to the stock, and that's the conditions between both here.

Similar, for the put you can derive these conditions, and again, that's because it is a heat equation. It turns out that for a simple derivative such that the calls and puts, it is possible to find an exact analytic solution. Here are exactimated set of solutions for a call, put and the digital contracts. Well, not surprising again, I mean they're all connected to the error function, so to the normal distribution, basically, as the

solutions of heat equation ought to be. Why do they look exactly the same? If we have five minutes at the end, we'll probably shed some light on the specific form of equations. But just trust that we can see that it's discounted, and what I'm saying is expected value of our payout on the [? risk ?] [UNINTELLIGIBLE] for measure.

Here is an example -- it's of a particular call option on the same IBM stock. So I chose the short-dated contract, just avoid the dividend payment. So it's a contract expiring on March 18. So there is some based expiration. The stock, as we saw, the expirations of stock, as we saw, is what's trading at 81.14. The volatility is somewhere around 14%, estimated either from other options or from historically. Here is the price of our contract. I also have a simple Black-Scholes calculator here, and let's see if we can match this price. So let's see, I believe the volatility was 13%, right, 13.47. The interest rate, it's already here. As we all know the Fed just bumped the interest rate, so at 4.75% right now. The strike of our option was 80 times expiration was actually 10 days, and this should be measured at a fraction of year. So we divide 10 by 365. This stock was trading at 81.14, if I'm not mistaken. Here is the price of our call options contract, which is 150. Well, it's within the offer.

So maybe our volatility's slightly off and if increase at, let's say, 2.14% it will go slightly up. 152. Well, in general, let's play a little bit with it. Well, it is very short-dated option, so the value of our option is very close to the payout. So if we increase the time to maturity, let's make it three years just to see where, so now what if our option is -- well, that's what it is. If increase volatility, sure enough, let's make it 30%. So what do we expect? We expect if volatility higher, then certain things higher, so the value of our contract should go up, and it sure does. So basically that's how Black-Scholes works.

[? Plane ?] [? Hills ?] does contract trade on the market, but unfortunately not all of these contracts are so simple as calls and puts. First of all, there are many more complicated products with more difficult payout, which will constitute different and [UNINTELLIGIBLE] will discontinue final conditions on our Black-Scholes equation. Moreover, we made an assumption that the volatility is constant with time, and interest rate is constant with time. It is certainly not true for the real world. Volatility

probably should be time dependent, and this would make the coefficients in our Black-Scholes equation time dependent. Unfortunately, these cannot be solved analytically.

So in most of the cases in practice, we will have to use some kind of numerical solution. Finding different methods is the typical approach for the heat equation. As you know, both [UNINTELLIGIBLE PHRASE] schemes, and you will discuss some of those in 18.086. Tree methods. Tree methods meaning that we go back to our one step tree, and basically assume that our time expiration is many [UNINTELLIGIBLE] time steps away and will grow the tree further, so from this node we have some more nodes, and so and so forth. That would imply the final condition at the end, and discount back using risk-neutral probabilities, and get the price down. So those are called tree methods. One can show that actually those tree methods are equivalent to find a difference -- expect to find a difference scheme. Those are very popular. But again, in tree methods, what is very important is to set the probability from your tree the transition probability was the right one. Since the right one risk-neutral probability. Probabilities implied by the market, actually.

Another important numerical method is Monte Carlo simulation where you would simulate many different scenarios of the development of your stock after the maturity. Then basically using this path you will find the expected value of your payout. But again, in order for this expected value to be the same as the risk-neutral value, as the arbitrage-free value, you have to develop your Monte Carlo simulations is risk-neutral probability. So, risk-neutral valuation is extremely important.

Here is actually the general risk-neutral statement, which one can prove is that actually the value of any derivative is just discounted expected value of the payout of this derivative at maturity, but you have to take this expectation and divide measure. Using the right measure, meaning that you have to set correctly the transition probability -- you have to make them market-neutral. Under this measure, actually the dynamics of our stocks looks slightly different, and as you can see, our drift becomes the interest rate.

So either [UNINTELLIGIBLE PHRASE] you measure, everything grows with our risk-free interest rate. Just to shed a little bit of light on how we go at the solutions for calls and puts, Black Shoals solutions for call and put, well this is the distribution of our stock, the normal distribution of our stock at time t , and if we take this distribution and integrate our payout of our co-option against this distribution -- in other words, find the expected value of payout of our full option under the risk-neutral measure, then sure enough you will get [UNINTELLIGIBLE PHRASE]. This illustrates the best because what is digital -- digital is just the probability to end up at both a strike at time t , right? So if you integrate this lognormal pdf from the strike k to the infinite, that will be your answer. This is a good exercise in integration to make sure that it's correct.

So let's see. To conclude, what we've seen. So, we have seen that more than derivatives business makes use of quite advanced mathematics, and what kinds of mathematics is used there. Well, partial differential equations are used heavily. Numerical methods for the solution of this partial differential equations are naturally used. In order to get the integrations, we actually need to operate in some stochastic calculus, meaning that we need to know how to deal with either calculus, either formula, Gaussian theorem, and so on and so forth.

The other thing is to be able to build simulations to solve the heat equation and all other equations that you might encounter. The topic which we didn't touch upon is statistics because, of course, very advanced statistics is used for many, many things for analyzing historical data, which can be quite beautiful for trading strategies and many others. Besides these five topics, there is much, much more to mathematical finance, which makes it a very, very exciting field to work in. That's what I wanted to talk about.

Thank you very much for your attention.

PROFESSOR: Maybe I'll ask a firm question about boundary conditions, because you had said that those are different for different contracts and how do you deal with them in the finite differences or the tree model or whatever. What would be a one typical one?

GUEST SPEAKER: Well, typical one -- two very typical ones. So those you basically make a grid of your problem, in particular you build a tree, which is actually a grid of all possible outcomes. You set them up as the ends, so your tree grows. So you set your boundaries here at the end, and well you set probably some initial -- this is final condition, so you set some boundary conditions here. So this is your time t . This is t zero times t -- this is zero, this is 1, this is 2, this is t . So you set your payout here, so it will be maximum of s minus k and zero.

PROFESSOR: How many time steps might you take in this?

GUEST SPEAKER: Well, you would do like daily for three months -- you do three month options.

PROFESSOR: Maybe 100 steps.

GUEST SPEAKER: Yeah, something like that. Well, if it's two year option, that you probably would do it weekly or something like that.

PROFESSOR: You don't get into large, what would be scientifically, large-scale calculating.

PROFESSOR: No, in finance you usually don't keep this problem--.

PROFESSOR: In finite differences, do you use like higher order as opposed, where you had second derivatives, would you always use second differences or second order accuracy?

GUEST SPEAKER: In general, yes. In general, second order accuracy. In general you don't go higher. I mean the precision -- well, it's within tenths, right. So you can not do better than that. So it depends -- well, it depends what kind of amount you are dealing with. If you're actually selling and buying units of stock, you might consider something more precise. But it's very problem-defined. So that's how we deal with it.

PROFESSOR: Any questions? We can put the mic on if you have a question.

STUDENT: [UNINTELLIGIBLE PHRASE].

GUEST SPEAKER: Well, it is market proc-- Yes. I mean this is just a numerical solution. So yeah, it is

market process. and basically all stochastic calculus is about market process, continuous market process.

PROFESSOR: Is the mathematics that you get involved with pretty well set now or is there a need for more mathematics, if I can ask the question that way?

PROFESSOR: Yeah, well in this field it is probably quite well set. But if you get into more complicated fields, especially into credit modeling, the model for the credits of certain companies, then mathematics is not quite set, because there you start talking about jump processes and [UNINTELLIGIBLE] Weiner processes, not just lognormal processes. This stochastic differential equations become very hard, but may be analytically practical. So from this point of view there is -- but it's not a fundamental mathematics, it's not that you are opening a new field, but definitely trying to solve a stochastic differential equation, which usually boils down to solving a partial differential equation analytically. Can be pretty hard in mathematical problem, view as a mathematical problem.

PROFESSOR: So you showed the example of Black-Scholes solver. Everybody has that available all the time?

GUEST SPEAKER: Oh yeah. On Chicago trading floor, the traders have calculators where they just press a button and it's just hard-wired there.

PROFESSOR: And they're printing out error functions, basically -- a combination of error function, yeah.

GUEST SPEAKER: Well, sure enough, nobody uses just -- I mean this was their approximate example and that's why I chose such short-dated stock that before it pays any dividends, and where we cannot the volatility is constant, and so on and so forth, match the prices. Otherwise, the prices wouldn't match.

PROFESSOR: Thank you.