5.6 Nonlinear Flow and Conservation Laws

Nature is nonlinear. The coefficients in the equation *depend on the solution* u. In place of $u_t = c\,u_x$ we will study $u_t + uu_x = 0$ and more generally $u_t + f(u)_x = 0$. These are "conservation laws" and the conserved quantity is the integral of u.

The first part of this book emphasized the **balance equation**: forces balance and currents balance. For steady flow this was Kirchhoff's Current Law: *flow in equals flow out*. The net flow was zero. Now the flow is *unsteady*—the "mass inside" is changing. So a new $\partial/\partial t$ term will enter the conservation law.

There is "flux" through the boundaries. In words, **the rate of change of mass inside a region equals that incoming flux**. For an interval $[a, b]$, the incoming flux is the difference in fluxes at the endpoints a and b:

Integral form $$\frac{d}{dt}\int_a^b u(x,t)\,dx = \boldsymbol{f(u(a,t)) - f(u(b,t))}\,. \tag{1}$$

In applications, u can be a *density* (of cars along a highway). The integral of u gives the mass (number of cars) between a and b. This number changes with time, as cars flow in at point a and out at point b. The flux is **density $\boldsymbol{u}$ times velocity $\boldsymbol{v}$**.

The integral form is fundamental. We can get a differential form by allowing b to approach a. Suppose $b - a = \Delta x$. If $u(x,t)$ is a smooth function, its integral over a distance Δx will have leading term $\Delta x\, u(a,t)$. So if we divide equation (1) by Δx, the limit as Δx approaches zero is $\partial u/\partial t = -\partial f(u)/\partial x$:

Differential form $$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \frac{\partial u}{\partial t} + f'(u)\,\frac{\partial u}{\partial x} = 0 \tag{2}$$

When $f(u) =$ density u times velocity $v(u)$, we can solve this single conservation law. For traffic flow, the velocity $v(u)$ can be measured (it will decrease as density increases). In gas dynamics there are also conservation laws for momentum and energy. The velocity v becomes another unknown, along with the pressure p. The **Euler equations** for gas dynamics in one space dimension include two additional equations:

Conservation of momentum $$\frac{\partial}{\partial t}(uv) + \frac{\partial}{\partial x}(uv^2 + p) = 0 \tag{3}$$

Conservation of energy $$\frac{\partial}{\partial t}(E) + \frac{\partial}{\partial x}(Ev + Ep) = 0\,. \tag{4}$$

Systems of conservation laws are more complicated, but our scalar equation (2) already has the possibility of **shocks**. A shock is a discontinuity in the solution $u(x,t)$, where the differential form breaks down and we need the integral form (1).

The other outstanding example, together with traffic flow, is **Burger's equation**, for $u =$ velocity. **The flux $f(u)$ is $\frac{1}{2}u^2$**. The "inviscid" form has no u_{xx}:

Burger's equation $$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0\,.$$

When both the density and velocity are unknowns, these examples combine into *conservation of mass and conservation of momentum*. Typically we change density to ρ. For small disturbances of a uniform density ρ_0, we could linearize the conservation laws and reach the wave equation (Problem). But the Euler and Navier-Stokes equations are truly nonlinear, and we begin the task of solving them.

We will approach conservation laws (and these examples) in three ways:

1. By following characteristics until trouble arrives: they separate or collide

2. By a special formula ( )

3. By finite difference and finite volume methods, which are the practical choice.

Characteristics

The one-way wave equation $u_t = c\,u_x$ is solved by $u(x,t) = u(x+ct, 0)$. Every initial value u_0 is carried along a characteristic line $x + ct = x_0$. Those lines are parallel when the velocity c is a constant.

The conservation law $u_t = +u\,u_x = 0$ will be solved by $u(x,t) = u(x - ut, 0)$. Every initial value $u_0 = u(x_0, 0)$ is carried along a characteristic line $\boldsymbol{x - u_0 t = x_0}$. Those lines are *not* parallel because their slopes depend on the initial value u_0.

Notice that the formula $u(x,t) = u(x - ut, 0)$ involves u on both sides. It gives the solution "implicitly." If the initial function is $u(x,0) = 1 - x$, for example, the formula must be solved for u:

$$u = 1 - (x - ut) \quad \text{gives} \quad (1-t)u = 1 - x \quad \text{and} \quad u = \frac{1-x}{1-t}\,. \tag{5}$$

This does solve Burger's equation, since the time derivative $u_t = (1-x)/(1-t)^2$ is equal to $-uu_x$. The characteristic lines (with different slopes) can meet. This is an extreme example, where all characteristics meet at the same point:

$$x - u_0 t = x_0 \quad \text{or} \quad x - (1 - x_0)t = x_0 \quad \text{which goes through } x = 1, t = 1 \tag{6}$$

You see how the solution $u = (1-x)/(1-t)$ becomes $0/0$ at that point $x = 1, t = 1$. Beyond their meeting point, the characteristics cannot completely decide $u(x,t)$.

A more fundamental example is the **Riemann problem**, which starts from two constant values $u = A$ and $u = B$. Everything depends on whether $A > B$ or $A < B$. On the left side of Figure 5.13, with $A > B$, *the characteristics meet*. On the right side, with $A < B$, *the characteristics separate*. Both cases present a new (nonlinear) problem, when we don't have a single characteristic that is safely carrying the correct initial value to the point. This Riemann problem has *two* characteristics through the point, or *none*:

Shock Characteristics *collide* (light goes red: speed drops from 60 to 0)
Fan Characteristics *separate* (light goes green: speed up from 0 to 60)

The problem is how to connect $u = 60$ to $u = 0$, when the characteristics don't give the answer. A shock will be sharp breaking (drivers only see the car ahead in this model). A fan will be gradual acceleration.

TO DO...

Figure 5.13: A shock when characteristics collide, a fan when they separate.

For the conservation law $u_t + f(u)_x = 0$, the characteristics are $x - f'(u_0)t = x_0$. That line has the right slope to carry the constant value $u = u_0$:

$$\frac{d}{dt}u(x_0 + St, t) = S\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \quad \text{when} \quad S = f'(u)\,. \tag{7}$$

The solution until trouble arrives is $u(x,t) = u(x - f'(u)t, 0)$.

Shocks

After trouble arrives, it will be the *integral form* that guides the choice of the correct solution u. If there is a jump in u (*a shock*), that integral from tells where the jump must occur. Suppose u has different values u_L and u_R at points x_L and x_R on the left and right sides of the shock:

Integral form $$\frac{d}{dt}\int_{x_L}^{x_R} u\,dx + f(u_R) - f(u_L) = 0\,. \tag{8}$$

If the position of the shock is $x = X(t)$, we take x_L and x_R very close to X. The values of $u(x,t)$ inside the integral are close to the constants u_L and u_R:

$$\frac{d}{dt}\big[(x - x_L)\,u_L + (x_R - X)\,u_R\big] + f(u_R) - f(u_L) \approx 0\,.$$

This gives the speed $s = dX/dt$ of the shock curve:

Jump condition $$s\,u_L - s\,u_R + f(u_R) - f(u_L) = 0$$

shock speed $$= \frac{f(u_R) - f(u_L)}{u_R - u_L} = \frac{[\,f\,]}{[\,u\,]}\,. \tag{9}$$

For the Riemann problem, the left and right values u_L and u_R will be constants A and B. The shock speed s is the ratio between the jump $[\,f\,] = f(B) - f(A)$ and the jump $[\,u\,] = B - A$. Since this ration gives a constant slope, the shock line is straight. For other problems, the characteristics are carrying different values of u into the shock. So the shock speed s is not constant and the shock lines is curved.

The shock gives the solution when characteristics collide. With $f(u) = \frac{1}{2}u^2$ in Burger's equation, the shock speed is halfway between u_L and u_R:

Burger's equation $$\text{Shock speed } s = \frac{1}{2}\frac{u_R^2 - u_L^2}{u_R - u_L} = \frac{1}{2}(u_R + u_L)\,. \tag{10}$$

The Riemann problem has $u_L = A$ and $u_R = B$, and s is their average. Figure 5.14 shows how the integral form of Burger's equation is solved by the right placement of the shock.

Fans

You might expect a similar picture (just flipped) when $A < B$. *Wrong*. The integral form is still satisfied, but it is also satisfied by a **fan**. The choice between shock and fan is made by the "**entropy condition**" that as t increases, *characteristics must go into the shock*. The wave speed is faster than the shock speed on the left, and slower on the right:

Entropy condition $$f'(u) > s > f'(u_R) \tag{11}$$

Since Burger's equation has $f'(u) = u$, it only has shocks when u_L is larger than u_R. In the Riemann problem that means $A > B$. In the opposite case, the smaller value $u_L = A$ has to be connected to $u_R = B$ by the fan in Figure 5.14:

Fan (or rarefaction) $$u = \frac{x}{t} \quad \text{for} \quad At < x < Bt\,. \tag{12}$$

use fig 6.28 p. 592 of IAM (reverse left and right figs)

Figure 5.14: Characteristics collide in a shock and separate in a fan.

Notice especially that in the traffic flow problem, the velocity $v(u)$ *decreases* as the density u increases. A good model is linear between $v = v_{\mathsf{max}}$ at zero density and $v = 0$ at maximum density. Then the flux $f(u) = u\,v(u)$ is a downward parabola (concave instead of Burger's convex $u^2/2$):

Traffic speed and flux $$v(u) = v_{\mathsf{max}}\left(1 - \frac{u}{u_{\mathsf{max}}}\right) \quad \text{and} \quad f(u) = v_{\mathsf{max}}\left(u - \frac{u^2}{u_{\mathsf{max}}}\right). \tag{13}$$

Typical values for a single lane of traffic show a maximum flux of $f = 1600$ vehicles per hour, when the density is $u = 80$ vehicles per mile. This maximum flow rate is attained when the velocity f/u is $v = 20$ miles per hour! Small comfort at that speed, to know that other cars are getting somewhere too.

Problems ____ and ____ compute the solution when a light goes red (shock travels backward) and when a light goes green (fan moves forward). Please look at the figures, to see how the vehicle trajectories are entirely different form the characteristics.

A driver keeps adjusting the density to stay safely behind the car in front. (Hitting the car would give $u < 0$.) We all recognize the frustration of braking and accelerating from a series of shocks and fans. This traffic crawl happens when the green light is too short for the shock to make it through.

A Solution Formula for Burger's Equation

Let me comment on three nonlinear equations. They are useful models, quite special because each one has an exact solution formula:

Conservation law	$u_t + u\,u_x = 0$
Burger's with viscosity	$u_t + u\,u_x = \nu\,u_{xx}$
Korteweg-de Vries	$u_t + u\,u_x = -a\,u_{xxx}$

The conservation law can develop shocks. This won't happen in the second equation because the u_{xx} viscosity term prevents it. That term can stay small when the solution is smooth, but it dominates when a wave is about to break. The profile is steep but it stays smooth.

As starting function for the conservation law, I will pick a point source: $u(x,0) = \delta(x)$. We can guess a solution with a shock, and check the jump condition and entropy condition. Then we find an exact formula when $\nu\,u_{xx}$ is included, by a neat change of variables that produces $h_t = \nu\,h_{xx}$. When we let $\nu \to 0$, the limiting formula solves the conservation law—and we can check that the following solution is correct.

Solution with $u(x,0) = \delta(x)$ When $u(x,0)$ jumps upward, we expect a fan. When it drops we expect a shock. The delta function is an extreme case (very big jumps up and down, very close together!). So we look for a shock curve $x = X(t)$ immediately in front of a fan!

Expected solution $$u(x,t) = \frac{x}{t} \quad \text{for} \quad 0 \le x \le X(t); \quad \text{otherwise } u = 0. \qquad (14)$$

The total mass at the start is $\int \delta(x)\,dx = 1$. This never changes, and already that locates the shock position $X(t)$:

$$\text{Mass at time } t = \int_0^X \frac{x}{t}\,dt = \frac{X^2}{2t} = 1 \qquad \text{so } X(t) = \sqrt{2t}\,. \qquad (15)$$

Does the drop in u, from $X/t = \sqrt{2t}/t$ to zero, satisfy the jump condition?

$$\text{Shock speed } s = \frac{dX}{dt} = \frac{\sqrt{2}}{2\sqrt{t}} \quad \text{equals} \quad \frac{\text{Jump } [\,u^2/2\,]}{\text{Jump } [\,u\,]} = \frac{X^2/2t^2}{X/t} = \frac{\sqrt{2t}}{2t}\,.$$

The entropy condition $u_L > s > u_R = 0$ is also satisfied, and the solution ( ) looks good. It *is* good, but because of the delta function we check it another way.

Begin with $u_t + u\,u_x = \nu\,u_{xx}$, and solve that equation exactly. If $u(x)$ is $\partial U/\partial x$, then integrating our equation gives $U_t + \frac{1}{2}U_x^2 = \nu\,U_{xx}$. The initial value $U_0(x)$ is now a step function. Then the great change of variables $U = -2\nu \log h$ produces the heat equation $h_t = \nu\,h_{xx}$ (Problem ____). The initial value becomes $h(x,0) = e^{-U_0(x)/2\nu}$. Section 5.4 found the solution to the heat equation $u_t = u_{xx}$ from any starting function $h(x,0)$ and we just change t to νt:

$$U(x,t) = -2\nu \log h(x,t) = -2\nu \log\left[\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty} e^{-U_0(y)/2\nu}\, e^{-(x-y)^2/4\nu t}\, dy\right]. \tag{16}$$

It doesn't look easy to let $\nu \to 0$, but it can be done. That exponential has the form $e^{-B(x,y)/2\nu}$. This is largest when B is smallest. An asymptotic method called "steepest descent" shows that as $\nu \to 0$, the bracketed quantity in (16) approaches $c\,e^{-B-\text{min}/2\nu}$. Taking its logarithm and multiplying by -2ν, (16) becomes $U = B_{\text{min}}$ in the limit:

$$\lim_{\nu\to 0} U(x,t) = B_{\text{min}} = \min_y \left[U_0(y) + \frac{1}{2t}(x-y)^2\right]. \tag{17}$$

This is the solution formula for $U_t + \frac{1}{2}U_x^2 = 0$. Its derivative $u = U_x$ solves the conservation law $u_t + u\,u_x = 0$. By including the viscosity $\nu\,u_{xx}$ with $\nu \to 0$, we are finding the $u(x,t)$ that satisfies the jump condition and the entropy condition.

Example Starting from $u(x,0) = \delta(x)$, its integral U_0 is a step function. The minimum of B is either at $y = x$ or at $y = 0$. Check each case:

$$U(t,x) = B_{\text{min}} = min_y \left[\begin{matrix} 0 & (y \le 0) \\ 1 & (y > 0) \end{matrix} \;+\; \frac{(x-y)^2}{2t}\right] = \begin{cases} 0 & \text{for } x \le 0 \\ x^2/2t & \text{for } 0 \le x \le \sqrt{2t} \\ 1 & \text{for } x \ge \sqrt{2t} \end{cases}$$

The result $u = dU/dx$ is $\mathbf{0}$ or $\boldsymbol{x/t}$ or $\mathbf{0}$. This agrees with our guess in equation ( )—a fan rising from 0 and a shock back to 0 at $x = \sqrt{2t}$.