18.085 Computational Science and Engineering I
Fall 2008

18.085 Quiz 3 October 7, 2008 Professor Strang

Your PRINTED name is ________________________ Student Number ________ Grading
1
2
3
4
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1. (25 points)

   (a) The 2π-periodic function $F(x)$ equals 1 for $0 \leq x < \pi$ and equals 0 for $\pi \leq x < 2\pi$. Find its Fourier coefficients c_k using complex exponentials:

   $$F(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}$$

   Write out the terms for $k = -1, 0, 1$. What is the decay rate of the c_k as $k \to \infty$? How do you see this from the function $F(x)$?

   (b) The energy equality connects $\int |F(x)|^2 \, dx$ with $\sum |c_k|^2$. What is this equation for our particular $F(x)$? *Find a formula for* π.

   (c) What is the derivative of this $F(x)$? *Draw the graph of* dF/dx *!!* What is the complex Fourier series for dF/dx? What is the decay rate of the coefficients? WHY?

2. (25 points) I am looking for the 7th degree polynomial $p(z) = c_0 + c_1 z + \cdots + c_7 z^7$ that has values $1, 0, 1, 0, 1, 0, 1, 0$ at the 8 points $z = 1,\ z = w,\ \ldots,\ z = w^7$. These points are the 8th roots of 1 with $w = e^{2\pi i/8}$ and $w^2 = e^{2\pi i/4} = i$.

(a) We have 8 equations for the 8 c's. The zeroth equation is $p(z) = 1$ at $z = 1$:

$$c_0 + c_1 + \cdots + c_7 = 1$$

What are the next two equations (at $z = w$ and $z = w^2$)? If you put all 8 equations in matrix form $A\boldsymbol{c}=\boldsymbol{b}$, *describe the matrix* A and the vector $\boldsymbol{b}$.

(b) By knowing the inverse matrix, you can solve those equations. *Write down* $\mathbf{c}= A^{-1}\boldsymbol{b}$. What is that inverse matrix?

(c) Now multiply to find the 8 components of $\boldsymbol{c}$. What is the polynomial $p(z)$? Please check that it has the right values $1, 0, 1, 0, 1, 0, 1, 0$ at $z = 1, w, \cdots, w^7$.

3. (25 points)

(a) Find the Fourier integral transform $\widehat{f}(k)$ of this function $f(x)$:

$$f(x) = 0 \text{ for } x < 0, \; f(x) = e^{-ax} \text{ for } x \geq 0$$

(b) Take the Fourier transform of each term in the differential equation:

$$\frac{du}{dx} + au(x) = \delta(x) \qquad -\infty < x < \infty$$

Now find $\widehat{u}(k)$. *Now find* $u(x)$.

(c) Check that your $u(x)$ does solve the differential equation at $x < 0$ and $x = 0$ and $x > 0$. If the right side of the equation changes from $\delta(x)$ to $\delta(x-1)$, find the new solution $U(x)$:

$$\frac{dU}{dx} + aU(x) = \delta(x-1)$$

4. (25 points)

(a) These matrix-vector multiplications Cx and Cy are the cyclic convolution of which vectors?

$$Cx = \begin{bmatrix} 5 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 \\ -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

and

$$Cy = \begin{bmatrix} 5 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 \\ -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 6 \\ -6 \end{bmatrix}$$

Take the Discrete Fourier Transform of all three vectors c, x, y. *Call those transforms* $\widehat{c}, \widehat{x}, \widehat{y}$.

(b) Convert those two cyclic convolutions Cx and Cy into component-by-component multiplications of the transforms. The answer uses numbers.

(c) Apparently this $y = (1, -1, 1, -1)$ is an eigenvector with $\lambda = 6$. Multiply any circulant matrix C times y to find the eigenvalue:

$$Cy = \begin{bmatrix} c_0 & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_3 & c_2 \\ c_2 & c_1 & c_0 & c_3 \\ c_3 & c_2 & c_1 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \lambda y.$$

How is λ connected to the transform $\widehat{c}$ of $c = (c_0, c_1, c_2, c_3)$?