18.085 Computational Science and Engineering I
Fall 2008

18.085 Quiz 1 October 7, 2008 Professor Strang

Your PRINTED name is ____________________ Student Number ________ Grading
1
2
3
4

1. Start with the equation $-\frac{d}{dx}(c(x)\frac{du}{dx}) = 1$. The fixed-fixed boundary conditions are $u(0) = 0 = u(1)$. The function $c(x)$ jumps from 1 to 2 at $x = \frac{1}{2}$:
$c(x) = 1$ for $x \leq \frac{1}{2}$ $\quad c(x) = 2$ for $x > \frac{1}{2}$.

   (a) Take $\triangle x = \frac{1}{4}$ and $u_0 = u_4 = 0$. Create a difference equation $A^TCAu = f$ that models this problem. What are the shapes of A and C? What are those matrices? Hint from review session: The FREE-FREE matrix is 4 by 5.

$$A_0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Solution: The fixed-fixed matrix A removes the boundary columns 1 and 5 of the free-free matrix A_0. So A is 4 by 3 and C is 4 by 4:

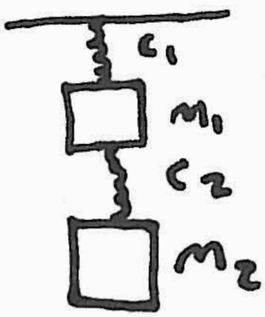
$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 2 \end{bmatrix}$$

   (b) Multiply A^TCA to find K. Circle one of these properties.
The matrix K is (positive definite) (only positive semidefinite) (indefinite)
Prove your statement *from the numbers in* K OR *from its form* $K = A^TCA$. Tell me which test for positive definiteness/semidefiniteness you are using.
Solution: The stiffness matrix A^TCA will be 3 by 3. It multiplies $u = (u_1, u_2, u_3)$.

$$K = A^TCA = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 4 \end{bmatrix}.$$

This is *positive definite.* All tests are passed. Upper left determinants are 2, then 5, then 12. Pivots are 2, then 5/2, then 12/5. Eigenvalues must be positive too. Energy $u^TA^TCAu = (Au)^TC(Au) = c_1e_1^2 + c_2e_2^2 + c_3e_3^2 + c_4e_4^2 > 0$.

2. (Two oscillating masses with fixed-free ends)



(a) Set up the matrix equations $M\frac{d^2u}{dt^2} + Ku = 0$ for this problem using masses m_1,m_2 and spring constants c_1,c_2. Find M and K.

Solution: The matrices are

$$M = \begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} \qquad K = A^TCA = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}$$

(b) What matrix eigenvalue problem for eigenvalues λ_1,λ_2 would you solve to find $u(t)$? What would be the form of $u(t)$ using the λ's and x's, with constants still to be determined by the initial conditions? NOT NECESSARY TO COMPUTE λ's and x's.

Solution: Solve $Kx = \lambda Mx$ to find the eigenvalues λ_1, λ_2 and eigenvectors x_1, x_2 of $M^{-1}K$.
Frequencies $\omega_1 = \sqrt{\lambda_1}$ and $\omega_2 = \sqrt{\lambda_2}$.
$u(t) = A(\cos\omega_1 t)x_1 + B(\sin\omega_1 t)x_1 + C(\cos\omega_2 t)x_2 + D(\sin\omega_2 t)x_2$.

3. Suppose we measure $b = 1, 3, 3$ at times $t = 0, 1, 2$. Those three points do not lie on a line $b = C + Dt$.

(a) Find the best C and D in the least-squares sense, to give the minimum error $E = e_1^2 + e_2^2 + e_3^2$. (The number e_3 is the error $C + 2D - 3$ at the third time $t = 2$.) SET UP THE MATRIX A AND THE LEAST SQUARES EQUATION AND SOLVE FOR C AND D.

Solution: The unsolvable equations $Au = b$ are

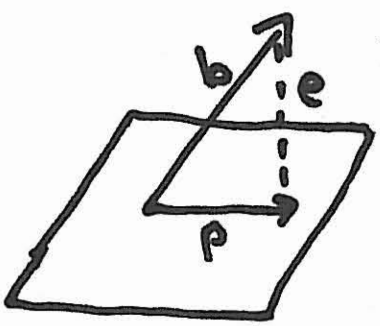
$$\begin{aligned} C + 0 \cdot D &= 1 \\ C + 1 \cdot D &= 3 \\ C + 2 \cdot D &= 3 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

The least squares equation $A^T A \hat{u} = A^T b$ is

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \qquad \text{and} \qquad \hat{u} = \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}.$$

Best line is $4/3 + t$.

(b) Geometrically, the vector $b = (1, 3, 3)$ is being projected onto some 2-dimensional plane. FIND THE PROJECTION $p = (p_1, p_2, p_3)$ AND THE ERROR $e = (e_1, e_2, e_3)$. If the measurements b had been the same as p, then the best line would have ________________ (Complete a suitable sentence).



Solution: The projection of b is

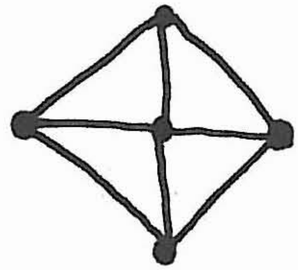
$$p = A\hat{u} = \begin{bmatrix} 4/3 \\ 7/3 \\ 10/3 \end{bmatrix}.$$

The error is

$$e = b - p = \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \end{bmatrix}.$$

[The problem statement allows $-e$ as a correct answer]
If $b = p$, then the best line would have –
"gone through the points",
"been the same line $4/3 + t$",
"..."

4. (a) What shape is the incidence matrix A for this graph?
How many independent columns in the matrix A?
Why is A^TA not invertible? Write two properties of A^TA.
NOT NECESSARY TO WRITE ANY MATRICES.



Solution: 8 edges, 5 nodes. The incidence matrix A is 8 by 5.
It has only 4 independent columns.
A^TA is not invertible because $u = (1,1,1,1,1)$ solves $Au = 0$ and then $A^TAu = 0$.

(b) I want to find the drop in the slope $\frac{du}{dx}$ at $x = \frac{1}{2}$, when

$$-\frac{d}{dx}(e^x \frac{du}{dx}) = \delta(x - \tfrac{1}{2}) \text{ with } u(0) = 0 \text{ and } u'(1) = 0.$$

Step 1 Solve $-\frac{dw}{dx} = \delta(x - \frac{1}{2})$ with $w(1) = 0$ to see the drop in w(x).
Step 2 Since $w(x) = e^x du/dx$, what is the drop in du/dx at $x = \frac{1}{2}$? Not necessary to find $u(x)$.

Solution: The solution to $-dw/dx = \delta(x-a)$ with $w(1) = 0$ is $w(x) = [1 \text{ for } x \le a, \text{ then } 0 \text{ for } x > a]$.
If $e^x du/dx = w(x)$, then the drop in du/dx will be e^{-a}.
This problem has $a = \frac{1}{2}$, so the drop is $1/\sqrt{e}$.
[Not difficult to solve for $u(x)$ in this fixed-free case!]