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18.085 Computational Science and Engineering I
Fall 2008

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18.085 Prof. Strang Quiz 3 Solutions December 3, 2003

Your name is: ______________________ **Grading** 1
2
3

Total

Thank you for taking 18.085, I hope you enjoyed it.

1) **(35 pts.)** Suppose the 2π-periodic $f(x)$ is a half-length square wave:

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \pi/2 \\ -1 & \text{for } -\pi/2 < x < 0 \\ 0 & \text{elsewhere in } [-\pi, \pi] \end{cases}$$

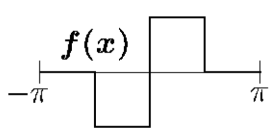
(a) Find the Fourier cosine and sine coefficients a_k and b_k of $f(x)$.

(b) Compute $\int_{-\pi}^{\pi} (f(x))^2\, dx$ as a number and also as an infinite series using the a_k^2 and b_k^2.

(c) DRAW A GRAPH of its integral $I(x)$. (Then $dI/dx = f(x)$ on the interval $[-\pi, \pi]$ choose the integration constant so $I(0) = 0$.) What are the Fourier coefficients A_k and B_k of $I(x)$?

(d) DRAW A GRAPH of the derivative $D(x) = \frac{df}{dx}$ from $-\pi$ to π. What are the Fourier coefficients of $D(x)$?

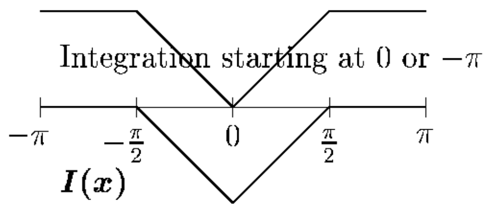
(e) If you convolve $D(x) * I(x)$ why do you get the same answer as $f(x) * f(x)$? Not required to find that answer, just explain $D * I = f * f$.

(a) 

$f(x) = \text{odd function} = -f(-x)$ so all $a_k = 0$.

Half-interval: $b_k = \frac{2}{\pi} \int_0^{\pi/2} \sin kx\, dx = \frac{2}{\pi} \frac{1 - \cos(k\pi/2)}{k}$.

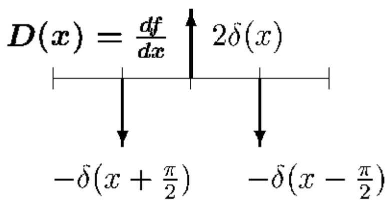
(b) $\int_{-\pi}^{\pi}(f(x))^2\,dx = \int_{-\pi/2}^{\pi/2} = \pi$. By Parseval this equals $\pi\sum b_k^2$. (Substituting $b_k = \frac{2}{\pi}\left(\frac{1}{1},\frac{2}{2},\frac{1}{3},\frac{0}{4},\ldots\right)$ will give a remarkable formula from $\sum b_k^2 = 1$.)

(c) 

Even function so $B_k = 0$.

Integrating $b_k \sin kx$ gives $-b_k \frac{\cos kx}{k}$ so $A_k = \frac{-b_k}{k}$.

The constant term is $A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} I(x)\,dx = \frac{3\pi}{8}$ or $-\frac{\pi}{8}$ (integrate starting at 0 or $-\pi$).

(d) 

Even function so $B_k = 0$.

Derivative of $b_k \sin kx$ is $kb_k \cos kx$ so $A_k = kb_k$.

Constant term is $A_0 = 0$.

(e) Convolution in x-space is multiplication in k-space. So $f * f$ has complex Fourier coefficients c_k^2 (with factor 2π). And $D(x) * I(x)$ has Fourier coefficients $(ikc_k)(c_k/ik) = c_k^2$ (with same factor). $D * I = f * f$!! Check in x-space:

$$\int_{-\pi}^{\pi} I(t)\,D(x-t)\,dt = \text{ integrate by parts } =$$

$$\int_{-\pi}^{\pi} f(t)\,f(x-t)\,dt + \text{ (boundary term } = 0 \text{ by periodicity)}.$$

The usual minus sign disappears because of 2nd minus sign: $\frac{d}{dt}D(x-t) = -f(x-t)$.

NOTE: I have now learned that we can't just multiply sine coefficients $(kb_k)(-b_k/k)$ because that gives an unwanted minus sign as in $\int \sin t \sin(x-t)\,dt = -\pi\cos x$.

2) **(33 pts.)** (a) Compute directly the convolution $f*f$ (cyclic convolution with $N=6$) when $f=(0,0,0,1,0,0)$. [You could connect vectors $(f_0,\ldots,f_5)$ with polynomials $f_0+f_1w+\cdots+f_5w^5$ if you want to.]

(b) What is the Discrete Fourier Transform $c=(c_0,c_1,c_2,c_3,c_4,c_5)$ of the vector $f=(0,0,0,1,0,0)$? Still $N=6$.

(c) Compute $f*f$ another way, by using c in "transform space" and then transforming back.

With $N=6$ the complex number $w=e^{2\pi i/6}$ has $w^3=-1$ and $\overline{w}^3=-1$ and $w^6=1$.

(a) $f=(0,0,0,1,0,0)$ corresponds to w^3. Then $f*f$ corresponds to w^6 which is 1. So $f*f=(1,0,0,0,0,0)$. (Also seen by circulant matrix multiplication.)

(b) The transform $c=F^{-1}f=\frac{1}{6}\overline{F}f=\frac{1}{6}$(column of $\overline{F}$ with powers of $\overline{w}^3=-1$): Then $c=\frac{1}{6}(1,-1,1,-1,1,-1)$.

(c) The transform of $f*f$ is $\frac{6}{36}(1^2,(-1)^2,1^2,(-1)^2,1^2,(-1)^2)=\frac{1}{6}(1,1,1,1,1,1)$.

Multiply that vector v by F to transform back and $Fv=(1,0,0,0,0,0)$ as in part (a)!

3) **(32 pts.)** On page 310 Example 3, the Fourier integral transform of the *one-sided* decaying pulse $f(x) = e^{-ax}$ (for $x \geq 0$) $f(x) = 0$ (for $x < 0$) is computed for $-\infty < k < \infty$ as

$$\widehat{f}(k) = \frac{1}{a + ik}\,.$$

(a) Suppose this one-sided pulse is shifted to start at $x = L > 0$:

$$f_L(x) = e^{-a(x-L)} \text{ for } x \geq L, \qquad f_L(x) = 0 \text{ for } x < L\,.$$

Find the Fourier integral transform $\widehat{f}_L(k)$.

(b) Draw a rough graph of the difference $D(x) = f(x) - f_L(x)$, on the whole line $-\infty < x < \infty$. Find its transform $\widehat{D}(k)$. NOW LET $a \to 0$.

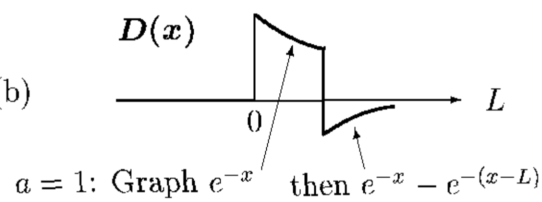
What is the limit of $D(x)$ as $a \to 0$?

What is the limit of $\widehat{D}(k)$ as $a \to 0$?

(c) The function $f_L(x)$ is smooth except for a <u>jump</u> at $x = L$, so the decay rate of $\widehat{f}_L(k)$ is <u>$1/k$</u>. The convolution $C(x) = f_L(x) * f_L(x)$ has transform $\widehat{C}(k) =$ <u>$e^{-i2kL}/(a+ik)^2$</u> with decay rate <u>$1/k^2$</u>. Then in x-space this convolution $C(x)$ has a <u>corner (= ramp)</u> at the point $x =$ <u>$2L$</u>.

(a) $f_L(x)$ is $f(x - L)$. By the shift rule (page 317) $\widehat{f}_L(k) = e^{-ikL}\widehat{f}(k) = \frac{e^{-ikL}}{a+ik}$.

(b)



As $a \to 0$, $D(x)$ approaches 1 for $0 < x < L$, 0 elsewhere

$\widehat{D}(k) = \frac{1}{a+ik} - \frac{e^{-ikL}}{a+ik}$ approaches $\frac{1-e^{-ikl}}{ik}$ = transform of square pulse.

(c) FILLED IN BLANKS ABOVE