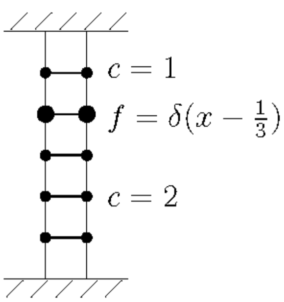
18.085 Computational Science and Engineering I
Fall 2008

18.085 Quiz 2 November 14, 2005 Professor Strang

Your PRINTED name is: ____________________

Grading **1**
2
3

1) **(34 pts.)** A point load at $x = \frac{1}{3}$ hangs at the same point where $c(x)$ changes from $c = 1$ (for $0 < x < \frac{1}{3}$) to $c = 2$ (for $\frac{1}{3} < x < 1$). Both ends are FIXED.



(a) Solve for $u(x)$ and $w(x) = c(x)\, u'(x)$:

$$-\frac{d}{dx}\left(c(x)\,\frac{du}{dx}\right) = \delta\left(x - \frac{1}{3}\right) \quad \text{with} \quad u(0) = u(1) = 0\,.$$

(b) Draw the graphs of $u(x)$ and $w(x)$.

(c) Divide the hanging bar into intervals of length $h = \frac{1}{6}$ (then $c(x)$ changes from 1 to 2 at $x = 2h$). There are unknowns $U = (u_1, \ldots, u_5)$ at the meshpoints. Write down a matrix approximation $\boldsymbol{KU} = \boldsymbol{F}$ to the equation above. Take differences of differences (each difference over an interval of length h).

Solution.

(a) $-dw/dx = \delta(x - 1/3)$ gives $w(x) = \begin{cases} B & x < 1/3 \\ B-1 & x > 1/3 \end{cases}$

Then $c(x)\,u'(x) = w(x)$ gives (including $c(x) = 1$ or 2)

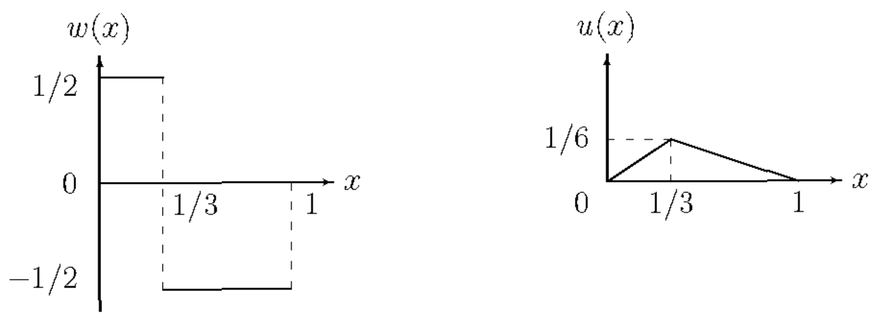
$$u(x) = \begin{cases} B\,x & x < 1/3 \\ \dfrac{B}{3} + \dfrac{(B-1)}{2}(x - 1/3) & x > 1/3 \end{cases}$$

$$u(1) = 0 \quad \text{gives} \quad \frac{B}{3} + \frac{(B-1)}{2}(x - 1/3) = 0 \longrightarrow B = 1/2$$

With $B = 1/2$ we have

$$w(x) = \begin{cases} 1/2 & x < 1/3 \\ -1/2 & x > 1/3 \end{cases} \qquad\qquad u(x) = \begin{cases} x/2 & x < 1/3 \\ -x/4 + 1/4 & x > 1/3 \end{cases}$$

(b)



(c) Finite difference approximation $KU = F$

$w = c(x)\,dw/dx$ is approximated by

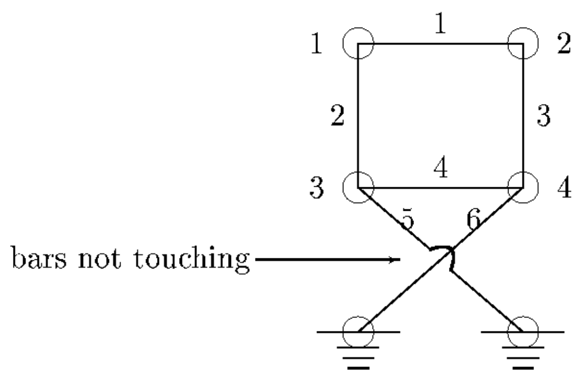
$$\frac{u_1-0}{h}\,,\ \frac{u_2-u_1}{h}\,,\ 2\,\frac{u_3-u_2}{h}\,,\ 2\frac{u_4-u_3}{h}\,,\ 2\frac{u_5-u_4}{h}\,,\ 2\frac{0-u_5}{h}\,.$$

Then $-dw/dx$ is approximated by

$$\frac{u_1-0}{h^2} - \frac{u_2-u_1}{h^2}\,,\ \frac{u_2-u_1}{h^2} - 2\,\frac{u_3-u_2}{h^2}\,,\ 2\,\frac{u_3-u_2}{h^2} - 2\frac{u_4-u_3}{h^2}\,,$$
$$2\frac{u_4-u_3}{h^2} - 2\frac{u_5-u_4}{h^2}\,,\ 2\frac{u_5-u_4}{h^2} - 2\frac{0-u_5}{h^2}\,.$$

$$K = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 3 & -2 & 0 & 0 \\ 0 & -2 & 4 & -2 & 0 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & -2 & 4 \end{bmatrix} \qquad F = \frac{1}{h} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2) **(33 pts.)** This truss doesn't look safe to me. Those angles are 45°. The matrix A will be 6 by 8 when the displacements are fixed to zero at the bottom.



(a) How many independent solutions to $e = Au = 0$? Draw these mechanisms.

(b) Write numerical vectors $u = (u_1^{\mathrm{H}}, u_1^{\mathrm{V}}, \ldots, u_4^{\mathrm{H}}, u_4^{\mathrm{V}})$ that solve $Au = 0$ to give those mechanisms in part (a).

(c) What is the first row of $A^{\mathrm{T}}A$ (asking about $A^{\mathrm{T}}A$!) if unknowns are taken in that usual order used in part (b) ?

Solution.

(a) $m = 6$ and $n = 8 \longrightarrow$ 2 mechanisms

(b) Numerical vectors for the mechanisms:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \qquad \textit{or} \text{ a combination of these two.}$$

(c) First row of $A^{\mathrm{T}}A = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

3) **(33 pts.)** (a) Is the vector field $w(x, y) = (x^2 - y^2, 2xy)$ equal to the gradient of any function $u(x)$? What is the divergence of w? If $u(x, y)$ and $s(x, y)$ are a Cauchy-Riemann pair, show that $w(x, y) = (s(x, y), u(x, y))$ will be a gradient field and also have divergence zero.

(b) Take real and imaginary parts of $f(x + iy) = (x + iy + \frac{1}{x+iy})$ to find two solutions of Laplace's equation. Write those two solutions also in polar coordinates.

(c) Integrate each of the functions $u = 1, u = r\cos\theta, u = r^2\cos 2\theta$ around the closed circle of radius 1 to find $\int u\, d\theta$. How could this same computation come from the Divergence Theorem?

Solution.

(a) i) $\operatorname{curl} w = \partial(2xy)/\partial x - \partial(x^2 - y^2)/\partial y = 4y \neq 0 \longrightarrow w(x,y)$ is not equal to the gradient of any function $u(x,y)$.

ii) $\operatorname{div} w = \partial(x^2 - y^2)/\partial x + \partial(2xy)/\partial y = 4x$.

Parts i) and ii) with $w = (u, s)$ were not gradients or divergence-free. Parts iii) and iv) have $w = (s, u)$ and this succeeds! For the function $f(x,y) = u(x,y) + is(x,y)$, Caucy-Riemann pair $u_x = s_y$, $u_y = -s_x$ and $w(x,y) = (s,u)$.

iii) $\operatorname{curl} w = u_x - s_y = 0$.

iv) $\operatorname{div} w = s_x + u_y = 0$.

(b) $f(x+iy) = x + iy + \dfrac{1}{x+iy} = x + iy + \dfrac{x-iy}{x^2+y^2} = x + \dfrac{x}{x^2+y^2} + i\left(y - \dfrac{y}{x^2+y^2}\right)$

$u = \operatorname{Re}(f) = x + \dfrac{x}{x^2+y^2} = \left(r + \dfrac{1}{r}\right) \cos\theta$

$s = \operatorname{Im}(f) = y - \dfrac{y}{x^2+y^2} = \left(r - \dfrac{1}{r}\right) \sin\theta$

(c) i) **Case 1:** $v = \operatorname{grad} u$ is not related to w and $\operatorname{div} w = 0$

$$\begin{aligned}
\iint_R (v_1 w_1 + v_2 w_2)\, dx\, dy &= \iint_R (\operatorname{grad} u) \cdot w\, dx\, dy \\
&= -\iint_R (\operatorname{div} w)\, u\, dx\, dy + \oint_B u\,(w \cdot n)\, ds \\
&= \oint_B u\,(w \cdot n)\, ds = 5 \iint_R (\operatorname{div} w)\, dx\, dy = 0\,.
\end{aligned}$$

ii) **Case 2:** $v = w$ and $\operatorname{div} w = 0$

From i), $\displaystyle\iint_R (v_1^2 + v_2^2)\, dx\, dy = 0.$

Since $v_1^2, v_2^2 \geq 0 \longrightarrow v_1^2 + v_2^2 = 0 \longrightarrow v_1 = v_2 = 0 \longrightarrow v = (0,0)$.