18.085 Computational Science and Engineering I
Fall 2008

# 18.085 Quiz 1 October 4, 2004 Professor Strang

**Your name is:** ______________________________ **Grading** 1.
2.
3.
4.

OPEN BOOK EXAM

Write solutions onto these pages!

Circles around short answers please!!

1) **(32 pts.)** This problem is about the symmetric matrix

$$H = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

(a) By elimination find the triangular $L$ and diagonal $D$ in $H = LDL^{\mathrm{T}}$.

(b) What is the smallest number $q$ that could replace the corner entry $H_{33} = 1$ and still leave $H$ positive *semi*definite? $q =$ _____

(c) $H$ comes from the 3-step framework for a hanging line of springs:

$$\text{displacements} \xrightarrow{A} \text{elongations} \xrightarrow{C} \text{spring forces} \xrightarrow{A^{\mathrm{T}}} \text{external force } f$$

What are the specific matrices $A$ and $C$ in $H = A^{\mathrm{T}}CA$?

(d) What are the requirements on any $m$ by $n$ matrix $A$ and any symmetric matrix $C$ for $A^{\mathrm{T}}CA$ to be positive definite?

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2) **(24 pts.)** Suppose we make three measurements $b_1, b_2, b_3$ at times $t_1, t_2, t_3$. They would fit exactly on a straight line $b = C + Dt$ if we could solve

$$\begin{aligned} C &+ Dt_1 &= b_1 \\ C &+ Dt_2 &= b_2 \\ C &+ Dt_3 &= b_3\,. \end{aligned}$$

(a) If $b_1, b_2, b_3$ are equally reliable **what are the equations** for the best values $\widehat{C}$ and $\widehat{D}$? *Don't solve the equations—write them in terms of t's and b's* (not just some letter $A$).

(b) Suppose the errors in $b_1, b_2, b_3$ are independent with variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$ (covariances = 0 because independent). **Find the new equations for the best $\widehat{C}$ and $\widehat{D}$. Use** $t$'s, $b$'s **and** $c_i = 1/\sigma_i^2$—by method ① or ②

① Remember how the covariance matrix $\Sigma$ (called $V$ in the book) enters the equations

② Divide the three equations above by $\sigma_1, \sigma_2, \sigma_3$. Then do ordinary least squares because the rescaled errors have variances = 1.

(c) Suppose $\sigma_1 = 1, \sigma_2 = 1$, but $\sigma_3 \to \infty$ so the third measurement is (*exactly reliable*) (*totally unreliable*) CROSS OUT ONE.

In this case the best straight line goes through which points?

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3) **(20 pts.)** (a) Suppose $\frac{du}{dx}$ is approximated by a *centered* difference:

$$\frac{du}{dx} \approx \frac{u(x+\Delta x) - u(x-\Delta x)}{2\,\Delta x}$$

With equally spaced points $x = h, 2h, 3h, 4h, 5h = \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$ and zero boundary conditions $u_0 = u_6 = 0$, *write down* the 3 by 3 centered first difference matrix $\Delta$:

$$(\Delta u)_i = \frac{u_{i+1} - u_{i-1}}{2h}\,.$$

Show that this matrix $\Delta$ is *singular* (because 3 is odd) by solving $\Delta u = 0$.

(b) Removing the last row and column of a positive definite matrix $K$ always leaves a positive definite matrix $L$. Why? Explain using one of the tests for positive definiteness.

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4) **(24 pts.)** The equation to solve is $-u'' + u = \delta(x - \frac{1}{2})$ with a unit point load at $x = \frac{1}{2}$ and zero boundary conditions $u(0) = u(1) = 0$.

(a) Solve $-u'' - u = 0$ starting from $x = 0$ with $u(0) = 0$. There will be one arbitrary constant $A$. Replace $x$ by $1 - x$ in your answer, to solve $-u'' - u = 0$ ending at $u(1) = 0$ with arbitrary constant $B$.

(b) Use the "jump conditions" at $x = \frac{1}{2}$ to find $A$ and $B$.

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