18.085 Computational Science and Engineering I
Fall 2008

18085 FALL 2002 EXAM 1 SOLUTIONS

DETAILED SOLUTIONS

Problem 1

Let $x = (t_1, ..., t_m)$ and $b = (b_1, ..., b_m)$. Then

$$A = \begin{bmatrix} 1 & t_1 \\ ... & ... \\ 1 & t_m \end{bmatrix};\ b = \begin{bmatrix} b_1 \\ ... \\ b_m \end{bmatrix}$$

and we would like to solve (but can't)

$$\begin{bmatrix} 1 & t_1 \\ ... & ... \\ 1 & t_m \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} b_1 \\ ... \\ b_m \end{bmatrix}$$

The short answer is that $A^T A$ is always positive definite. The long answer is that A is of rank 1 only if all $t_1 = ... = t_m$ (all measurements taken at the same time and the m points lie on one vertical line). In that extreme case $A^T A$ is positive semidefinite.

We have for the matrix $A^T A$:

$$\begin{aligned} A^T A &= \begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} \\ A^T b &= \begin{bmatrix} \sum b_i \\ \sum b_i t_i \end{bmatrix} \end{aligned}$$

If the center of mass is at the origin (i.e. $\sum t_i = \sum b_i = 0$) then

$$\begin{aligned} A^T A &= \begin{bmatrix} m & 0 \\ 0 & \sum t_i^2 \end{bmatrix} \\ A^T b &= \begin{bmatrix} 0 \\ \sum b_i t_i \end{bmatrix} \end{aligned}$$

ehen we solve the resulting system

$$\begin{bmatrix} m & 0 \\ 0 & \sum t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ \sum b_i t_i \end{bmatrix}$$

we get

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\sum b_i t_i}{\sum t_i^2} \end{bmatrix}$$

The fact that $C = 0$ indicates that the least squares line $b = Dt$ passes through the origin.

For all points to lie on the same line we must have one of the following

1. (from geometric considerations)

$$\frac{b_2 - b_1}{t_2 - t_1} = \frac{b_3 - b_2}{t_3 - t_2}$$

2. b needs to be in the column space of A. From the columns of A this is equivalent to the fact that there exist two numbers c and d such that

$$b_i = c + dt_i$$

But then

$$\frac{b_i - b_j}{t_i - t_j} = \frac{dt_i - dt_j}{t_i - t_j} = d$$

and we once again have condition 1.

3. Perform Gaussian elimination

$$\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & t_1 \\ 0 & t_2 - t_1 \\ 0 & t_3 - t_1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - b_1 \\ b_3 - b_1 \end{bmatrix}$$

In order for this system to be solvable, we must have

$$\begin{bmatrix} t_2 - t_1 \\ t_3 - t_1 \end{bmatrix} d = \begin{bmatrix} b_2 - b_1 \\ b_3 - b_1 \end{bmatrix}$$

which is once again equivalent to 1.

4. The matrix $[A\ b]$ is singular:

$$[A\ b] = \begin{bmatrix} 1 & t_1 & b_1 \\ 1 & t_2 & b_2 \\ 1 & t_3 & b_3 \end{bmatrix}$$

The determinant of this matrix must equal 0 and since

$$\begin{aligned} \det [A\ b] &= b_2 t_1 - b_1 t_2 + b_1 t_3 - b_3 t_1 - b_2 t_3 + b_3 t_2 \\ &= (b_3 - b_1)(t_2 - t_1) - (b_2 - b_1)(t_3 - t_1) \end{aligned}$$

we once again obtain a condition equivalent to 1.

Problem 2

Let's point the coordinate axis down. The matrix A is 3×2:

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

Then the stiffness matrix K is given by

$$\begin{aligned} K &= A^T C A \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & S \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -1 & S+1 \end{bmatrix} \end{aligned}$$

and the resulting system becomes

$$\begin{bmatrix} 2 & -1 \\ -1 & S+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} mg \\ mg \end{bmatrix}$$

Therefore,

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= -\begin{bmatrix} 2 & -1 \\ -1 & S+1 \end{bmatrix}^{-1} \begin{bmatrix} mg \\ mg \end{bmatrix} \\ &= \frac{1}{2S+1} \begin{bmatrix} S+1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} mg \\ mg \end{bmatrix} \\ &= mg \begin{bmatrix} \frac{S+2}{2S+1} \\ \frac{3}{2S+1} \end{bmatrix} \end{aligned}$$

For $S = \infty$, we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = mg \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

and for $S = 0$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = mg \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

As for the tensions, we have

$$\begin{aligned} y &= CAx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & S \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} x \\ &= mg \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & S \end{bmatrix} \begin{bmatrix} \frac{S+2}{2S+1} \\ \frac{3}{2S+1} \end{bmatrix} = \\ &= mg \begin{bmatrix} \frac{S+2}{2S+1} \\ -\frac{S-1}{2S+1} \\ -\frac{3S}{2S+1} \end{bmatrix} \end{aligned}$$

Therefore, when $S = \infty$

$$y = mg \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$$

and when $S = 0$

$$y = mg \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Note: when it comes to finding the limiting values of y you should not take the limit until the very last moment!

Problem 3

Point each arrow from the lower numbered node to the higher. Then A is 8×5:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

When nodes 6 and 7 are grounded, equation $Ax = 0$ has no non-zero solutions. $A^T y = 0$, on the other hand, has as many independent solutions as there are loop. Since both nodes 6 and 7 are grounded there are effectively 3 loops: $(1, 2, 3)$, $(3, 4, 5, 6)$, and $(6, 7, 8)$.

Since there are no batteries, the following system will give us potentials x:

$$A^T C A x = -f$$

or (forming $A^T C A$ was not required)

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 3 \end{bmatrix} x = - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then y is given by

$$y = -CAx$$

Solving for x (not required) we get

$$x = - \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \\ 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Then y is (not required):

$$y = -\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} \\ 1 \\ 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

and this can definitely be guessed by just looking at the picture.

Now let's treat the picture as a truss. It has 10 degrees of movement (2 at each node) and 8 edges. The truss matrix A is 8×10.

The truss has two mechanisms. 1. The top 5 nodes all move to one side. 2. The top triangle moves rigidly to one side. Further, any combination of these two mechanisms is also a mechanism, but not an independent one.

For the remaining part pick the most conventional Cartesian coordinate system and order the degrees of freedom in the usual way. Then the two mechanisms are given by

$$x_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \; x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(a) Since A is not full rank, $K = A^T A$ is only positive semidefinite.

(b) The general answer is that f need to be orthogonal to both x_0 and x_1. A particular example of this is any collection of vertical forces. Or else equal and opposite horizontal forces on nodes 2 and 3 (or nodes 4 and 5).