18.085 Computational Science and Engineering I
Fall 2008

18.085 - Mathematical Methods for Engineers I	Prof. Gilbert Strang

Solutions - Problem Set 7

3.4, Problem 4. Show that $u = r\cos\theta + r^{-1}\cos\theta$ solves Laplace's equation (13), and express u in terms of x and y. Find $v = (u_x, u_y)$ and verify that $v \cdot n = 0$ on the circle $x^2 + y^2 = 1$. This is the velocity of flow past a circle in Figure 3.18.

<u>Show $u = r\cos\theta + r^{-1}\cos\theta$ solves Laplace's equation</u>:

Laplace's equation in r, θ is:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{13}$$

Now, if $u = r\cos\theta + r^{-1}\cos\theta$, then

$$\begin{aligned} \frac{\partial u}{\partial r} &= \cos\theta - r^{-2}\cos\theta & \frac{\partial u}{\partial \theta} &= -r\sin\theta - r^{-1}\sin\theta \\ \frac{\partial^2 u}{\partial r^2} &= 2r^{-3}\cos\theta & \frac{\partial^2 u}{\partial \theta^2} &= -r\cos\theta - r^{-1}\cos\theta. \end{aligned}$$

Plugging this into the left-hand side of (13), we get

$$2r^{-3}\cos\theta + \frac{1}{r}(\cos\theta - r^{-2}\cos\theta) + \frac{1}{r^2}(-r\cos\theta - r^{-1}\cos\theta) = (2-1-1)r^{-3}\cos\theta + (1-1)\frac{1}{r}\cos\theta = 0,$$

as desired.

<u>U in (x, y)</u>:

Now, since $x = r\cos\theta, y = r\sin\theta \Rightarrow r = \sqrt{x^2+y^2}, \theta = \tan^{-1}\left(\frac{y}{x}\right)$. So

$$u = r\cos\theta + r^{-1}\cos\theta = x + \frac{x}{x^2+y^2} = \frac{x(1+x^2+y^2)}{x^2+y^2}.$$

So

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1 + \frac{(x^2+y^2)\cdot 1 - x(2x)}{(x^2+y^2)^2} = 1 + \frac{y^2-x^2}{(x^2+y^2)^2}. \\ \frac{\partial u}{\partial y} &= 0 + x(-1\cdot(x^2+y^2)^{-2}\cdot 2y) = \frac{-2xy}{(x^2+y^2)^2}. \end{aligned}$$

So

$$v = \left(1 + \frac{y^2-x^2}{(x^2+y^2)^2}, \frac{-2xy}{(x^2+y^2)^2}\right).$$

<u>$v \cdot n = 0$</u>:

$$\begin{aligned} n &= (x, y) \text{ is normal to the circle} \\ v &= (1 + y^2 - x^2, -2xy) \text{ on the circle} \\ v \cdot n &= x - x^3 - xy^2 = x - x(x^2+y^2) = 0 \text{ on the circle} \end{aligned}$$

3.4, Problem 17. Verify that $u_k(x,y) = \frac{\sin(\pi kx)\sinh(\pi ky)}{\sinh(\pi k)}$ solves Laplace's equation for $k = 1, 2, \ldots$. Its boundary values on the unit square are $u_0 = \sin(\pi kx)$ along $y = 1$ and $u_b = 0$ on the three lower edges. Recall that $\sinh(x) = \frac{e^z - e^{-z}}{2}$.

$$u = u_k(x,y) = \frac{\sin(\pi kx)\cancel{\frac{1}{2}}(e^{\pi ky} - e^{-\pi ky})}{\cancel{\frac{1}{2}}(e^{\pi k} - e^{-\pi k})}.$$

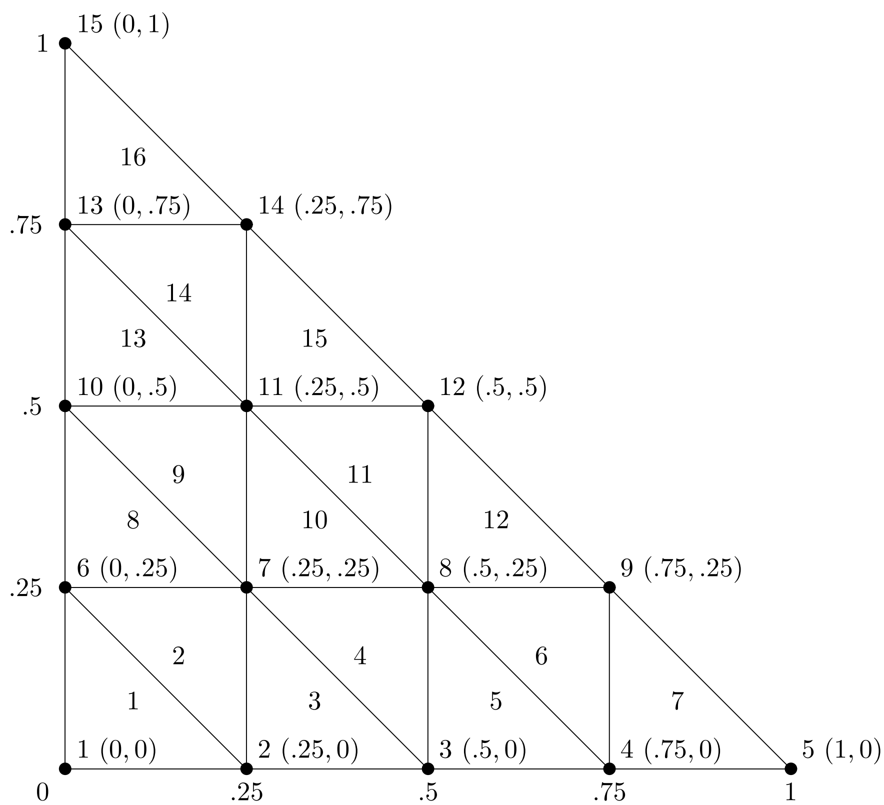
$$\frac{\partial u}{\partial x} = \pi k\cos(\pi kx)\cdot\frac{e^{\pi ky} - e^{-\pi ky}}{e^{\pi k} - e^{-\pi k}} \qquad \frac{\partial u}{\partial y} = \sin(\pi kx)\cdot\frac{\pi k e^{\pi ky} + \pi k e^{-\pi ky}}{e^{\pi k} - e^{-\pi k}}$$

$$\frac{\partial^2 u}{\partial x^2} = -\pi^2 k^2\sin(\pi kx)\cdot\frac{e^{\pi ky} - e^{-\pi ky}}{e^{\pi k} - e^{-\pi k}} \qquad \frac{\partial^2 u}{\partial y^2} = \sin(\pi kx)\cdot\frac{\pi^2 k^2(e^{\pi ky} - e^{-\pi ky})}{e^{\pi k} - e^{-\pi k}}$$

So

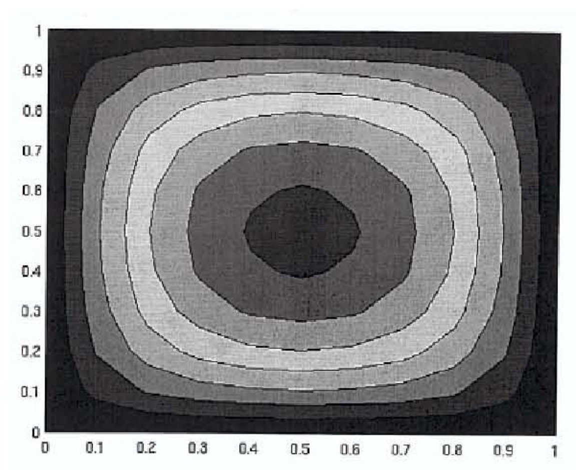
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

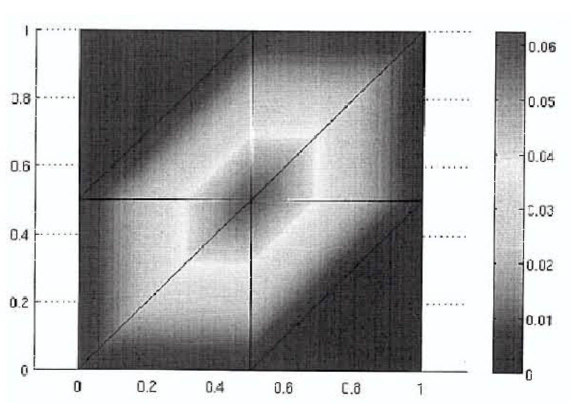
3.6, Problem 6. Solve Poisson's equation $-u_{xx} - u_{yy} = 1$ with $u = 0$ on the standard unit triangle T using Persson's code with $h = 0.25$. The mesh information is in the lists p, t, and b (boundary nodes) : 15 nodes, 16 triangles, 12 boundary nodes (nodes numbered by rows).



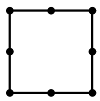
$$p = \begin{pmatrix} 0 & 0 \\ .25 & 0 \\ .5 & 0 \\ .75 & 0 \\ 1 & 0 \\ 0 & .25 \\ .25 & .25 \\ .5 & .25 \\ .75 & .25 \\ 0 & .5 \\ .25 & .5 \\ .5 & .5 \\ 0 & .75 \\ .25 & .75 \\ 0 & 1 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \end{matrix} \qquad t = \begin{pmatrix} 1 & 2 & 6 \\ 2 & 7 & 6 \\ 2 & 3 & 7 \\ 3 & 8 & 7 \\ 3 & 4 & 8 \\ 4 & 9 & 8 \\ 4 & 5 & 9 \\ 6 & 7 & 10 \\ 7 & 11 & 10 \\ 7 & 8 & 11 \\ 8 & 12 & 11 \\ 8 & 9 & 12 \\ 10 & 11 & 13 \\ 11 & 14 & 13 \\ 11 & 12 & 14 \\ 13 & 14 & 15 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \end{matrix} \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 9 \\ 10 \\ 12 \\ 13 \\ 14 \\ 15 \end{pmatrix}$$

contourf(xx,yy,zz);





3.6, Problem 9. The square drawn below has 8 nodes rather than the usual 9 for biquadratic Q_2. Therefore we remove the x^2y^2 term and keep



$$U = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^2y + a_8xy^2.$$

Find the $\phi(x, y)$ which equals 1 at $x = y = 0$ and zero at all other nodes.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \scriptstyle \leftarrow x=0,\ y=0 \\ \scriptstyle \leftarrow x=1,\ y=0 \\ \scriptstyle \leftarrow x=0,\ y=1 \\ \scriptstyle \leftarrow x=.5,\ y=0 \\ \scriptstyle \leftarrow x=0,\ y=.5 \\ \scriptstyle \leftarrow x=.5,\ y=1 \\ \scriptstyle \leftarrow x=1,\ y=.5 \\ \scriptstyle \leftarrow x=1,\ y=1 \end{array} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & .5 & 0 & .25 & 0 & 0 & 0 & 0 \\ 1 & 0 & .5 & 0 & 0 & .25 & 0 & 0 \\ 1 & .5 & 1 & .25 & .5 & 1 & .25 & .5 \\ 1 & 1 & .5 & 1 & .5 & .25 & .5 & .25 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}.$$

$$\phi \qquad = \qquad xy \qquad a$$

We solve for a to obtain

$$a = xy \backslash \phi = \begin{bmatrix} 1 \\ -3 \\ -3 \\ 2 \\ 5 \\ -2 \\ -2 \\ -2 \end{bmatrix}.$$

Alternatively, we can use Method B and set

$$\phi(x, y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^2y + a_8xy^2.$$

Then $1 - 3x + 2x^2 = \phi(x, 0) = a_1 + a_2x + a_4x^2$, so that $a_1 = 1, a_2 = -3, a_4 = 2$.

Likewise, $1 - 3y + 2y^2 = \phi(0, y) = a_1 + a_3y + a_6y^2$, so that $a_1 = 1, a_3 = -3, a_6 = 2$.

But $0 = \phi(x, 1) = a_1 + a_2x + a_3 + a_4x^2 + a_5x + a_6 + a_7x^2 + a_8x$, so that

$$\begin{aligned} a_1 + a_3 + a_6 &= 0 \\ a_2 + a_5 + a_8 &= 0 \\ a_4 + a_7 &= 0. \end{aligned}$$

From this, we can conclude that $a_7 = -2$ and $a_5 + a_8 = 3$.

Finally, $0 = \phi(1, y) = a_1 + a_2 + a_3y + a_4 + a_5y + a_6y^2 + a_7y + a_8y^2$, so that

$$\begin{aligned} a_1 + a_2 + a_4 &= 0 \\ a_3 + a_5 + a_7 &= 0 \\ a_6 + a_8 &= 0. \end{aligned}$$

From this, we know that $a_8 = -2$, $a_5 = 5$. So again,

$$a = \begin{bmatrix} 1 \\ -3 \\ -3 \\ 2 \\ 5 \\ -2 \\ -2 \\ -2 \end{bmatrix}.$$

3.6, Problem 16. Find the eigenvalues of $KU = \lambda MU$ if $K = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. This gives the oscillation frequencies for two unequal masses in a line of springs.

$$\begin{aligned} KU &= \lambda MU \\ M^{-1}KU &= \lambda U \\ \det(M^{-1}K - \lambda I) &= 0 \end{aligned}$$

$$M^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \qquad \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$M^{-1}K = \begin{bmatrix} 2 & -1 \\ -.5 & 1 \end{bmatrix} \qquad M^{-1}K = \begin{bmatrix} 2-\lambda & -1 \\ -.5 & 1-\lambda \end{bmatrix}$$

$$\begin{aligned} (2-\lambda)(1-\lambda) - (-1)(-.5) &= 0 \\ \lambda^2 - 3\lambda + 1.5 &= 0 \\ \lambda &= \frac{\sqrt{3}+3}{2} \text{ or } -\frac{\sqrt{3}-3}{2}. \end{aligned}$$