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18.085 Computational Science and Engineering I
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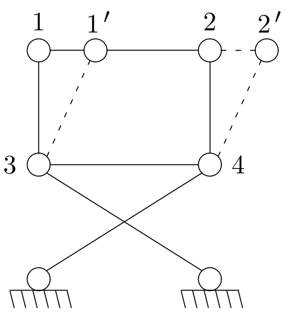
18.085 - Mathematical Methods for Engineers I Prof. Gilbert Strang

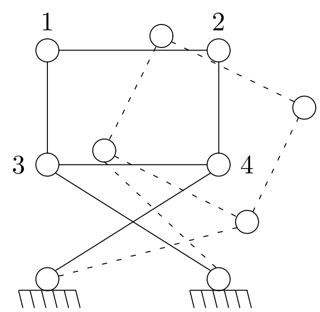
Solutions - Problem Set 5

Section 2.7, Problem 1: How many independent solutions to $Au = 0$. Draw them and find solutions $u = (u_1^H, u_1^V, \ldots, u_4^H, u_4^V)$. What shapes are A and $A^{\mathrm{T}}A$? First rows?

6 bars $\Rightarrow n = 2(6) - 4 = 8$ unknown disp. So $Au = 0$ has $8 - 6 = 2$ independent solutions (mechanisms).



$$u = \begin{pmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \\ u_3^H \\ u_3^V \\ u_4^H \\ u_4^V \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



$$u = \begin{pmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \\ u_3^H \\ u_3^V \\ u_4^H \\ u_4^V \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

$$A = 6 \times 8 \text{ matrix} \qquad A^{\mathrm{T}} = 8 \times 6 \text{ matrix} \qquad A^{\mathrm{T}}A = 8 \times 8 \text{ matrix}$$

First row of A:

$$[\ -1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\ ].$$

First row of $A^{\mathrm{T}}A$:

$$[\ 1 \quad 0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\ ].$$

Section 2.7, Problem 2: 7 bars, $N = 5$ so $n = 2N - r = 2(5) - 2 = 8$ unknown displacements.

a) What motion solves $Au = 0$?

b) By adding one bar, can A become square/invertible?

c) Write out row 2 of A (for bar 2 at 45° angle).

d) Third equation in $A^{\mathrm{T}}w = f$ with right side f_2^H?

Solution:

a) $8 - 7 = 1$ rigid motion, rotation about node 5.

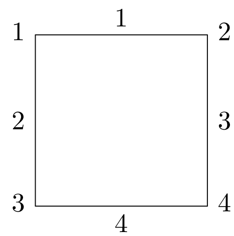
b) No $\rightarrow$ rigid motion only, so adding a bar won't get rid of motion. Needs another support.

c) $\left[\ 0 \quad 0 \quad -\cos\left(\frac{\pi}{2}\right) \quad \sin\left(\frac{\pi}{2}\right) \quad \cos\left(\frac{\pi}{2}\right) \quad -\sin\left(\frac{\pi}{2}\right) \quad 0 \quad 0\ \right] = \left[\ 0 \quad 0 \quad -\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad -\frac{\sqrt{2}}{2} \quad 0 \quad 0\ \right].$

d) $w_1 + f_2^H + w_4 \cos\theta + w_2 \cos\theta = 0.$

$$\boxed{-f_2^H = w_1 + w_4 \cos\theta - w_2 \cos\theta, \quad \cos\theta = 1/\sqrt{2}.}$$

Section 2.7, Problem 3:



a) Find $8-4$ independent solutions to $Au=0$.

b) Find 4 sets of f's so $A^{\mathrm{T}}w=f$ has a solution.

c) Check that $u^{\mathrm{T}}f=0$ for those four u's and f's.

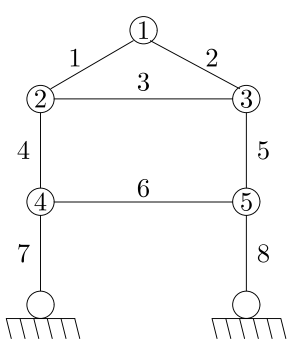
a) Solutions:

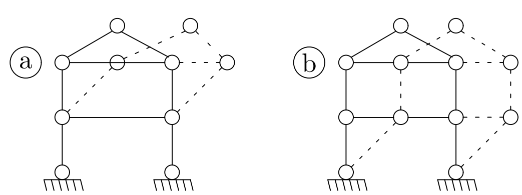
$$\begin{array}{lll} \text{horizontal:} & u_1 = & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \\ \text{vertical:} & u_2 = & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \\ \text{rotation (about node 3):} & u_3 = & \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \end{bmatrix} \\ \text{mechanism:} & u_4 = & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{array}$$

b) $$f_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad f_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad f_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad f_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

c) Each $u_i^{\mathrm{T}}f_j=0$

Section 2.7, Problem 5: Is $A^{\mathrm{T}}A$ positive definite? Semidefinite? Draw complete set of mechanisms.





8 bars, 7 nodes, 4 fixed displacements $(u_6^H, u_6^V, u_7^H, u_7^V)$. So $n=2(7)-4=10$, and $10-8=2$ solutions.

There are 2 solutions, therefore the truss is unstable → **not** positive definite.

$A^{\mathrm{T}}A$ must be positive semidefinite because A has dependent columns.

Section 3.1, Problem 1: Constant c, decreasing $f = 1 - x$, find $w(x)$ and $u(x)$ as in equations 9-10. Solve with $w(1) = 0, u(1) = 0$.

$$\begin{aligned}
w(x) &= -\int_0^x (1-s)ds + C_1 = \left(-x + \frac{x^2}{2}\right) + C_1 \\
w(1) &= 0 \Rightarrow -\left(1 - \frac{1}{2}\right) + C_1 = 0 \Rightarrow C_1 = \frac{1}{2} \\
\Rightarrow w(x) &= \frac{x^2}{2} - x + \frac{1}{2} \\
u(x) &= \int_0^x \frac{w(s)}{c(s)} ds = \frac{1}{c}\int_0^x \left(\frac{s^2}{2} - s + \frac{1}{2}\right) ds = \frac{1}{c}\left(\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{2}\right) + C_2.
\end{aligned}$$

Case 1: $u(0) = 0 \Rightarrow 0 + C_2 = 0 \Rightarrow C_2 = 0$.

$$\boxed{u(x) = \frac{1}{c}\left(\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{2}\right)}$$

Case 2: $u(1) = 0$ and $u(0) = 0$ (fixed, fixed).

$$\begin{aligned}
u(x) &= \frac{1}{c}\int_0^x \left(\frac{s^2}{2} - s + C_1\right) ds = \frac{1}{c}\left[\frac{s^3}{6} - \frac{s^2}{2} + C_1 s\right]_0^x \\
&= \frac{1}{c}\left(\frac{x^3}{6} - \frac{x^2}{2} + C_1 x + C_2\right) \\
u(0) = 0 &\Rightarrow C_2 = 0 \\
u(1) = 0 &\Rightarrow \frac{1}{c}\left(\frac{1}{6} - \frac{1}{2} + C_1\right) = 0 \Rightarrow C_1 = \frac{1}{3}
\end{aligned}$$

$$\boxed{u(x) = \frac{1}{c}\left(\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{3}\right)}$$

Section 3.1, Problem 5: $f =$ constant, c jumps from $c = 1$ for $x \le \frac{1}{2}$ to $c = 2$ for $x > \frac{1}{2}$. Solve $-\frac{dw}{dx} = f$ with $w(1) = 0$ as before, then solve $c\frac{du}{dx} = w$ with $u(0) = 0$.

$w(x) = \int_x^1 f dx = (1-x)f$

For $0 \le x \le \frac{1}{2}$, $\frac{\partial u}{\partial x} = (1-x)f, u(0) = 0 \rightarrow u(x) = \int_0^x (1-x) f dx = \left(x - \frac{x^2}{2}\right) f$. So $u\left(\frac{1}{2}\right) = \frac{3}{8} f$.

For $\frac{1}{2} \le x \le 1$, $\frac{\partial u}{\partial x} = (1-x)f, u\left(\frac{1}{2}\right) = \frac{3}{8}f \rightarrow u(x) = \frac{f}{2}\int_{1/2}^x (1-x) f dx + u\left(\frac{1}{2}\right) = \frac{7}{16} - \frac{1}{4} f(1-x)^2$.

In summary,

$$\boxed{u(x) = (1-x)f \qquad u(x) = \begin{cases} \left(x - \frac{x^2}{2}\right) f, & 0 \le x \le \frac{1}{2} \\ \frac{7}{16} f - \frac{1}{4} f (1-x)^2, & \frac{1}{2} \le x \le 1. \end{cases}}$$

Section 3.1, Problem 10: Use three hat functions with $h = \frac{1}{4}$ to solve $-u'' = 2$ with $u(0) = u(1) = 0$. Verify that the approximation U matches $u = x - x^2$ at the nodes.

1) $\int_0^1 c(x) \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx = \int_0^1 f(x)v(x)dx.$
2) $v_i = \phi_i$
3) Assume $c(x) = 1$.
4) Let $f(x) = 1$.

$$\left.\begin{array}{lllll} F_1 & = & \int_0^{1/2} 1 \cdot \phi_1(x)dx & = & \frac{1}{2}bh = \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4}. \\ F_2 & = & \int_{1/4}^{3/4} 1 \cdot \phi_2(x)dx & = & \frac{1}{2}bh = \frac{1}{4}. \\ F_3 & = & \int_{1/2}^{0} 1 \cdot \phi_3(x)dx & = & \frac{1}{2}bh = \frac{1}{4} \end{array}\right\} \text{right side}$$

$$\begin{array}{lllll} K_{11} & = & & & \int_0^{1/2} c(x)\frac{\partial \phi_1}{\partial x} \cdot \frac{\partial v_1}{\partial x} dx = \int_0^{1/4} 1 \cdot 1 \cdot 4dx + \int_{1/4}^{1/2} 1 \cdot (-4) \cdot 4 \; dx = 8. \\ K_{12} & = & K_{21} & = & \int_0^1 c(x)\frac{\partial \phi_2}{\partial x} \cdot \frac{\partial v_2}{\partial x} \; dx = \int_{1/4}^{1/2} 1 \cdot (-4) \cdot 4dx = -16\left(\frac{1}{4}\right) = -4. \\ K_{13} & = & K_{31} & = & \int_0^1 c(x)\frac{\partial \phi_1}{\partial x} \cdot \frac{\partial v_3}{\partial x} \; dx = 0. \end{array}$$

So $KU = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \end{bmatrix} = F.$

Solving yields $u = \begin{bmatrix} 0.1875 \\ 0.250 \\ 0.1875 \end{bmatrix}$. This approximation matches $u = x - x^2$ at the nodes since when $x = \begin{bmatrix} 0.25 \\ 0.50 \\ 0.75 \end{bmatrix}$, $u(x) = \begin{bmatrix} 0.1875 \\ 0.250 \\ 0.1875 \end{bmatrix}$, as desired.

Section 3.1, Problem 18: Fixed-free hanging bar $u(1) = 0$ is a natural boundary condition. To the N hat functions ϕ_i at interior meshpoints, add the half-hat that goes up to $U_{N+1} = 1$ at the endpoint $x = 1 = (N+1)h$. This $\phi_{N+1} = V_{N+1}$ has nonzero slope $\frac{1}{h}$.

a) The N by N stiffness matrix K for $-u_{xx}$ now has an extra row and column. How does the new last row of K_{N+1} represent $u'(1) = 0$?

$$\begin{bmatrix} \vdots & & & & & & \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{h} & \frac{1}{h} \end{bmatrix}.$$

$$u_{i-1} + 2u_i - u_{i+1} = f_i.$$

For row $n+1$, $-u_n + 2u_{n+1} - u_{n+2} = f_{n+1}$. If $u_{n+2} = u_{n+1}$, then the slope at $(n+1)$ is zero, and so we have $-u_n + u_{n+1} = f_{n+1}$.

b) For constant load, find the new last component $F_{N+1} = \int f_0 V_{N+1} dx$. Solve $K_{N+1}U = F$ and compare U with the true mesh values of $f_0\left(x - \frac{1}{2}x^2\right)$.

$$
\begin{aligned}
K_{11} &= \int c(x)\frac{\partial\phi_1}{\partial x}\cdot\frac{\partial v_1}{\partial x}dx = \int_0^{1/3} 1\cdot 3f_0\cdot 3\ dx + \int_{1/3}^{2/3} 1\cdot(-3)\cdot(-3)\ dx = 6. \\
K_{12} &= K_{21} = \int c(x)\frac{\partial\phi_1}{\partial x}\cdot\frac{\partial v_2}{\partial x}dx = \int_{1/3}^{2/3} 1\cdot(-3)\cdot(3)\ dx = -3 = K_{23} = K_{32}. \\
K_{13} &= K_{31} = 0. \qquad K_{33} = 3. \\
F_1 &= \int_0^1 \phi_1 f_0\ dx = \frac{1}{2}\frac{2}{3}f_0 = \frac{1}{3}f_0 \\
F_2 &= \int_0^1 \phi_2 f_0\ dx = \frac{1}{2}\frac{2}{3}f_0 = \frac{1}{3}f_0 \\
F_3 &= \int_0^1 \phi_3 f_0\ dx = \frac{1}{2}\frac{1}{3}f_0 = \frac{1}{6}f_0
\end{aligned}
$$

$$
\begin{aligned}
KU &= F \\
\begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} &= \begin{bmatrix} f_0/3 \\ f_0/3 \\ f_0/6 \end{bmatrix} \\
U &= \begin{bmatrix} 5/18 \\ 4/9 \\ 1/2 \end{bmatrix} f_0
\end{aligned}
$$

Indeed, considering $u = f_0\left(x - \frac{1}{2}x^2\right)$, we find that the values exactly match up:

$$
\begin{aligned}
x = \frac{1}{3} &\rightarrow \frac{5}{18}f_0 \\
x = \frac{2}{3} &\rightarrow \frac{4}{9}f_0 \\
x = 1 &\rightarrow \frac{1}{2}f_0.
\end{aligned}
$$