Exercises on similar matrices and Jordan form

Problem 28.1: (6.6 #12. *Introduction to Linear Algebra:* Strang) These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors; one from each block. However, their block sizes don't match and they are *not similar*:

$$J = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ and } K = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

For a generic matrix M, show that if $JM = MK$ then M is not invertible and so J is not similar to K.

Solution: Let $M = (m_{ij})$. Then:

$$JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } MK = \begin{bmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{bmatrix}$$

If $JM = MK$, then $m_{11} = m_{22} = 0$, $m_{21} = 0$, $m_{31} = m_{42} = 0$, and $m_{41} = 0$. Thus, the first column of M is all zeros and M is not invertible.

If J were similar to K there would be an invertible matrix M that satisfies $K = M^{-1}JM$, and so $MK = JM$. We just showed that there can be no such invertible matrix M. Therefore J is not similar to K.

Problem 28.2: (6.6 #20.) Why are these statements all true?

a) If A is similar to B then A^2 is similar to B^2.

b) A^2 and B^2 can be similar when A and B are not similar (try $\lambda = 0, 0$.)

c) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$.

d) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is not similar to $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.

e) Given a matrix A, let B be the matrix obtained by exchanging rows 1 and 2 of A and then exchanging columns 1 and 2 of A. Show that A is similar to B.

Solution:

a) If A is similar to B, then:

$$A = M^{-1}BM \Longrightarrow A^2 = M^{-1}BM(M^{-1}BM) = M^{-1}B^2M.$$

Since $A^2 = M^{-1}B^2M$, by definition A^2 is similar to B^2.

b) Let:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $A^2 = B^2$ and so A^2 is similar to B^2, but A is not similar to B because nothing but the zero matrix is similar to the zero matrix.

c) There are multiple ways to verify that the given matrices are similar. One way is to explicitly find the matrix M that satisfies the similarity condition:

$$\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

d) The given matrices are not similar because for the first matrix every vector in the plane is an eigenvector for $\lambda = 3$, whereas the second matrix only has a line ($y = 0$) of eigenvectors corresponding to $\lambda = 3$. We know that similar matrices have the same number of independent eigenvectors, so the given matrices cannot be similar.

e) To exchange the first two rows of A, we multiply A on the left by:

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

In order to exchange the first two columns of A, we multiply it on the right by the same matrix M. We thus have $B = MAM$. Since $M^{-1} = M$, B is similar to A.