

Exercises on linear transformations and their matrices

Problem 30.1: Consider the transformation T that doubles the distance between each point and the origin without changing the direction from the origin to the points. In polar coordinates this is described by

$$T(r, \theta) = (2r, \theta).$$

- Yes or no: is T a linear transformation?
- Describe T using Cartesian (xy) coordinates. Check your work by confirming that the transformation doubles the lengths of vectors.
- If your answer to (a) was "yes", find the matrix of T . If your answer to (a) was "no", explain why the T isn't linear.

Solution:

a) Yes. In terms of vectors, $T(\mathbf{v}) = 2\mathbf{v}$, so $T(\mathbf{v}_1 + \mathbf{v}_2) = 2\mathbf{v}_1 + 2\mathbf{v}_2 = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ and $T(c\mathbf{v}) = 2c\mathbf{v} = cT(\mathbf{v})$.

b) $T(x, y) = (2x, 2y)$. We know $\left| \begin{bmatrix} x \\ y \end{bmatrix} \right| = \sqrt{x^2 + y^2}$ and can calculate

$$\left| T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) \right| = \left| \begin{bmatrix} 2x \\ 2y \end{bmatrix} \right| = \sqrt{4(x^2 + y^2)} = 2 \left| \begin{bmatrix} x \\ y \end{bmatrix} \right|.$$

This confirms that T doubles the lengths of vectors.

c) We can use our answer to (b) to find that the matrix of T is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Problem 30.2: Describe a transformation which leaves the zero vector fixed but which is not a linear transformation.

Solution: If we limit ourselves to "simple" transformations, this is not an easy task!

One way to solve this is to find a transformation that acts differently on different parts of the plane. If $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ |y| \end{bmatrix}$ then

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

is not equal to

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Another approach is to use a nonlinear function to define the transformation: if $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y^2 \end{bmatrix}$ then $T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} cx \\ c^2y^2 \end{bmatrix}$ is not equal to $cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} cx \\ cy^2 \end{bmatrix}$.

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