

## 18.06SC Unit 3 Exam Solutions

- 1 (34 pts.) (a) If a square matrix  $A$  has all  $n$  of its *singular values* equal to 1 in the SVD, what basic classes of matrices does  $A$  belong to? (Singular, symmetric, orthogonal, positive definite or semidefinite, diagonal)

- (b) Suppose the (orthonormal) columns of  $H$  are eigenvectors of  $B$ :

$$H = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \quad H^{-1} = H^T$$

The eigenvalues of  $B$  are  $\lambda = 0, 1, 2, 3$ . Write  $B$  as the product of 3 specific matrices. Write  $C = (B + I)^{-1}$  as the product of 3 matrices.

- (c) Using the list in question (a), which basic classes of matrices do  $B$  and  $C$  belong to? (Separate question for  $B$  and  $C$ )

*Solution.*

(a) If  $\sigma = I$  then  $A = UV^T =$  product of orthogonal matrices = orthogonal matrix.

2nd proof: All  $\sigma_i = 1$  implies  $A^T A = I$ . So  $A$  is orthogonal.

( $A$  is *never* singular, and it won't always be symmetric — take  $U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  and  $V = I$ , for example. This also shows it can't be diagonal, or positive definite or semidefinite.)

(b)  $B = H\Lambda H^{-1}$  with  $\Lambda = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & 3 \end{bmatrix}$

$(B+I)^{-1} = H(\Lambda+I)^{-1}H^{-1}$  with (same eigenvectors)  $(\Lambda+I)^{-1} = \begin{bmatrix} 1 & & & \\ & 1/2 & & \\ & & 1/3 & \\ & & & 1/4 \end{bmatrix}$

(c)  $B$  is singular, symmetric, positive semidefinite.

$C$  is symmetric positive definite.

- 2 (33 pts.) (a) Find three eigenvalues of  $A$ , and an eigenvector matrix  $S$ :

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

- (b) Explain why  $A^{1001} = A$ . Is  $A^{1000} = I$ ? Find the three diagonal entries of  $e^{At}$ .

- (c) The matrix  $A^T A$  (for the same  $A$ ) is

$$A^T A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{bmatrix}.$$

How many eigenvalues of  $A^T A$  are positive? zero? negative? (Don't compute them but explain your answer.) Does  $A^T A$  have the same eigenvectors as  $A$ ?

*Solution.*

(a) The eigenvalues are  $-1, 0, 1$  since  $A$  is triangular.

$$\lambda = -1 \text{ has } x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \lambda = 0 \text{ has } x = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \lambda = 1 \text{ has } x = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix}.$$

Those vectors  $x$  are the columns of  $S$  (upper triangular!).

(b)  $A = \Lambda S^{-1}$  and  $A^{1001} = S \Lambda^{1001} S^{-1}$ . Notice  $\Lambda^{1001} = \Lambda$ ,  $A^{1000} = I$  ( $A$  is singular) ( $0^{1000} = 0 \neq 1$ ).

$e^{At}$  has  $e^{-1t}$ ,  $e^{0t} = 1$ ,  $e^t$  on its diagonal. *Proof using series:*

$\sum_0^\infty (At)^n/n!$  has triangular matrices so the diagonal has  $\sum (-t)^n/n! = e^{-t}$ ,  $\sum 0^n/n! = 1$ ,  $\sum t^n/n! = e^t$ .

*Proof using  $S \Lambda S^{-1}$ :*

$$e^{At} = S e^{\Lambda t} S^{-1} = \begin{bmatrix} 1 & \times & \times \\ 0 & 1 & \times \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & & \\ & 1 & \\ & & e^t \end{bmatrix} \begin{bmatrix} 1 & \times & \times \\ 0 & 1 & \times \\ 0 & 0 & 1 \end{bmatrix}.$$

(c)  $A^T A$  has 2 positive eigenvalues (it has rank 2, its eigenvalues can never be negative).

One eigenvalue is zero because  $A^T A$  is singular. And  $3 - 2 = 1$ .

(Or:  $A^T A$  is symmetric, so the eigenvalues have the same signs as the pivots.

Do elimination: the pivots are 1, 0, and  $42 - 16 = 26$ .)

**3 (33 pts.)** Suppose the  $n$  by  $n$  matrix  $A$  has  $n$  orthonormal eigenvectors  $q_1, \dots, q_n$  and  $n$  positive eigenvalues  $\lambda_1, \dots, \lambda_n$ . Thus  $Aq_j = \lambda_j q_j$ .

(a) What are the eigenvalues and eigenvectors of  $A^{-1}$ ? *Prove that your answer is correct.*

(b) Any vector  $b$  is a combination of the eigenvectors:

$$b = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n .$$

What is a quick formula for  $c_1$  using orthogonality of the  $q$ 's?

(c) The solution to  $Ax = b$  is also a combination of the eigenvectors:

$$A^{-1}b = d_1 q_1 + d_2 q_2 + \cdots + d_n q_n .$$

What is a quick formula for  $d_1$ ? You can use the  $c$ 's even if you didn't answer part (b).

*Solution.*

(a)  $A^{-1}$  has eigenvalues  $\frac{1}{\lambda_j}$  with the same eigenvectors

$$Aq_j = \lambda_j q_j \longrightarrow q_j = \lambda_j A^{-1} q_j \longrightarrow A^{-1} q_j = \frac{1}{\lambda_j} q_j.$$

(b) Multiply  $b = c_1 q_1 + \cdots + c_n q_n$  by  $q_1^T$ .

Orthogonality gives  $q_1^T b = c_1 q_1^T q_1$  so  $c_1 = \frac{q_1^T b}{q_1^T q_1} = q_1^T b$ .

(c) Multiplying  $b$  by  $A^{-1}$  will multiply each  $q_i$  by  $\frac{1}{\lambda_i}$  (part (a)). So  $c_i$  becomes

$$d_1 = \frac{c_1}{\lambda_1} \quad \left( = \frac{q_1^T b}{\lambda_1 q_1^T q_1} \text{ or } \frac{q_1^T b}{\lambda_1} \right).$$

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