

18.06 Linear Algebra, Fall 2011
Recitation Transcript – Exam #3 Problem Solving

DAVID SHIROKOFF: Hi, everyone. So for this problem, we're just going to take a look at computing some eigenvalues and eigenvectors of several matrices. And this is just a review problem for exam number three.

So specifically, we're given a projection matrix which has the form of $\frac{aa^T}{a^T a}$, where a is the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. The second problem is for a rotation matrix Q which is the numbers 0.6 , negative 0.8 , 0.8 , and 0.6 . And then the third one is for a reflection matrix which is $2P$ minus the identity. So I'll let you work these out. And then I'll come back in a second, and I'll fill in my solutions.

Hi, everyone. Welcome back. OK, so for the first problem, we're given a matrix P , which is a projection matrix. And from earlier on in the course, we probably already know that the eigenvalues of a projection matrix are either 0 or 1 .

And I'll just recall, how do you know that? Well if x is an eigenvector of P , then it satisfies the equation $Px = \lambda x$. But for a projection matrix, $P^2 = P$. So if P is a projection, we have $P^2 x = P x$.

And specifically, what this means is $P^2 x = \lambda x$. So we have P acting on $P x$ is equal to λx . And on the left hand side, $P x$ is going to give me a λx . $P x$ again will give me a λx . So we get $\lambda^2 x = \lambda x$.

And if I bring everything to the left hand side, I get $\lambda^2 x - \lambda x = 0$. And because x is not a zero vector, what that means is λ has to be either 0 or 1 . So this is just a quick proof that the eigenvalue of a projection matrix is either 0 or 1 .

So we already know that P is going to have eigenvalues of 0 or 1 . Now specifically, how do I identify which eigenvectors correspond to 0 and which eigenvectors correspond to 1 ? Well, in this case, P has a specific form, which is $\frac{aa^T}{a^T a}$.

So I'll just write out explicitly what this is. So a transpose a , 1 divided by a transpose a , is going to be 9 plus 16 on the denominator. Then we're going to have 3 and 4 and 3 and 4 .

Now when we have a matrix of this form, it's always going to be the case that the vector a is going to be an eigenvector with eigenvalue 1 . So let's check. What is P acting on a ?

Well, we end up with the matrix P is $\frac{1}{25} \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}$. This is the matrix P . And if we acted on the vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, notice how this piece right here, we can multiply out. This is going to be a transpose, and this is going to be a . And if we multiply these two pieces out, we get 25 , which is exactly the denominator a transpose a .

So at the end of the day, we get $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Because the 25 divides out with the 25 . Now note that this is exactly what we started with. This is exactly a . So note here that the vector a corresponds to an eigenvalue of 1 .

Meanwhile, for an eigenvalue of 0 , well, it always turns out to be the case that if I take any vector perpendicular to a , P acting on that vector is going to be 0 . So what's a vector, which I'll call b , that's perpendicular to a ?

Well, note that a is just a two by two vector. So that means there's only going to be one direction that's perpendicular to a . Now just by eyeballing it, I can see that a vector that's going to be perpendicular to a is negative 4 and 3 . So let's quickly check that this is an eigenvector of P with eigenvalue 0 . So what we need to show is that P acting on this vector, b , is 0 .

So P acting on b is going to be $\frac{1}{25}$. It's going to be $\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}$, multiplied by negative 4 , 3 . And note how when I multiply out this row on this column, I get negative 3 times 4 plus 3 times 4 , which is going to be 0 . OK? So this shows that this vector b has an eigenvalue of 0 because note that we can write this as $0b$.

OK. For the second part, Q , what are the eigenvectors and eigenvalues of this matrix, Q ? Well, Q is a rotation matrix. So I'll just write out Q again, 0.6 , negative 0.8 , 0.8 , 0.6 .

So note that we can identify the diagonal elements with a cosine of some angle θ . And we can associate the off diagonal parts as $\sin \theta$ and $-\sin \theta$. And the reason we can do that is because $0.6^2 + 0.8^2 = 1$. So this is a rotation matrix.

Now to work out the eigenvalues, I take a look at the characteristic equation. So this is going to give me, if I take a look at the characteristic equation, it's going to be $0.6 - \lambda$, squared. Then we have -0.8 times 0.8 . So that's going to be $+0.8$ squared. And we want this to be 0.

So if I rewrite this, I get $\lambda = 0.6 \pm 0.8i$, where i is the imaginary number. So notice how the eigenvalues come in complex conjugate pairs. And this is always the case when we have a real matrix.

So we can find, first off, just the eigenvalue that corresponds to $0.6 + 0.8i$. And then at the end, we'll be able to find the second eigenvector by just taking the complex conjugate of the first one. So let's compute $Q - \lambda I$.

And if we have this acting on some eigenvector, u , we want this to be 0. Now $Q - \lambda I$ is going to be, for the case $\lambda = 0.6 + 0.8i$, this is going to give me a quantity of $-0.8i$, -0.8 , 0.8 , and $-0.8i$. And I'm going to write down components of u , which are u_1 and u_2 . And we want this to vanish.

And we note that the second row is a constant multiple of the first row. Specifically, if I multiplied this first row through by i , we would get $-i^2$, which is just 1. And then the second part would be $-i$, so we would just get the second row back, which is good.

So we just need to find u_1, u_2 that are orthogonal to this first row. And again, just by inspection, I can pick 1 and $-i$. So note that that would give me $-0.8i + 0.8i$, and this vanishes. So this is the eigenvector that corresponds to the eigenvalue $\lambda = 0.6 + 0.8i$.

In the meantime, if I take the second eigenvalue, which is negative $0.8i$, I can take u , which is just the complex conjugate of this u up here. So it'll be $1 + i$. So this concludes the eigenvalues and eigenvectors of this matrix Q .

OK. Now lastly, number three, we're looking at a reflection matrix which has the form $2P - I$, where P is the same matrix that we had in part one. Now at first glance, it looks like we might have to diagonalize this entire matrix. However, note that by shifting $2P$ by I , we only shift the eigenvalues. And we don't actually change the eigenvectors. So note that this matrix R , which is $2P - I$, it's going to have the same eigenvectors as P . It's just going to have different eigenvalues.

So first off, we're going to have one eigenvector. So the first eigenvector is going to be a . So we have one eigenvector which is a . So we have one eigenvector which is a .

And note that for the vector a , it corresponds to the eigenvalue of 1 . So what eigenvalue does this correspond to? This is going to give me a λ which is $2 \times 1 - 1$. So it's 1 . So note that a , the vector a , not only has an eigenvalue of 1 for P , but it has an eigenvalue of 1 for R as well.

The second case was b . And remember that b has an eigenvalue of 0 for P . So when we act R acting on b , we'll have $2 \times 0 - 1b$. So this is going to give us negative b .

So the eigenvalue for b is going to be negative 1 . OK. And this is actually a general case for reflection matrices, is that they typically have eigenvalues of plus 1 or negative 1 .

OK, so we've just taken a look at several matrices that come up in practice. We've looked at projection matrices, reflection matrices, and rotation matrices. And we've seen a little bit of the properties of their eigenvalues and eigenvectors. So I'll just conclude here, and good luck on your test.

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