Exercises on complex matrices; fast Fourier transform

Problem 26.1: Compute the matrix F_2.

Solution: $F_2 = \begin{bmatrix} 1 & 1 \\ 1 & w \end{bmatrix}$, where $w = e^{i2\pi/2} = -1$. Hence

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Problem 26.2: Find the matrices D and P used in the factorization:

$$F_4 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} P$$

Hint: D is created using fourth roots, not square roots, of 1. Check your answer by multiplying.

Solution: We computed $F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ in the previous problem.
$D = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ contains the first two fourth roots of 1 (and $-D$ contains the others, -1 and -i).

P is a permutation matrix that arranges the components of the incoming vector so that its even components come first. For F_4, that means swapping the first and second components:

$$P \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_2 \\ x_1 \\ x_3 \end{bmatrix}.$$

$$\text{So, } P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Finally, we check our work by multiplying:

$$\begin{aligned}
\begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} P &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = F_4. \checkmark
\end{aligned}$$