Exercises on projection matrices and least squares

Problem 16.1: (4.3 #17. *Introduction to Linear Algebra:* Strang) Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1$, $b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\hat{\mathbf{x}} = (C, D)$ and draw the closest line.

Solution: $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$.

The solution $\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{9} \\ \mathbf{4} \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

Problem 16.2: (4.3 #18.) Find the projection $\mathbf{p} = A\hat{\mathbf{x}}$ in the previous problem. This gives the three heights of the closest line. Show that the error vector is $\mathbf{e} = (2, -6, 4)$. Why is $P\mathbf{e} = \mathbf{0}$?

Solution: $\mathbf{p} = A\hat{\mathbf{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\mathbf{b} - \mathbf{p} = (2, -6, 4)$. This error $\mathbf{e}$ has $P\mathbf{e} = P\mathbf{b} - P\mathbf{p} = \mathbf{p} - \mathbf{p} = \mathbf{0}$.

Problem 16.3: (4.3 #19.) Suppose the measurements at $t = -1, 1, 2$ are the errors 2, -6, 4 in the previous problem. Compute $\hat{\mathbf{x}}$ and the closest line to these new measurements. Explain the answer: $\mathbf{b} = (2, -6, 4)$ is perpendicular to _____ so the projection is $\mathbf{p} = \mathbf{0}$.

Solution: If $\mathbf{b} =$ error $\mathbf{e}$ then $\mathbf{b}$ is perpendicular to the column space of A. Projection $\mathbf{p} = \mathbf{0}$.

Problem 16.4: (4.3 #20.) Suppose the measurements at $t = -1, 1, 2$ are $\mathbf{b} = (5, 13, 17)$. Compute $\hat{\mathbf{x}}$ and the closest line and $\mathbf{e}$. The error is $\mathbf{e} = \mathbf{0}$ because this $\mathbf{b}$ is _____.

Solution: If $\mathbf{b} = A\hat{\mathbf{x}} = (5, 13, 17)$ then $\hat{\mathbf{x}} = (9, 4)$ and $\mathbf{e} = \mathbf{0}$ since $\mathbf{b}$ is *in the column space of* A.

Problem 16.5: (4.3 #21.) Which of the four subspaces contains the error vector $\mathbf{e}$? Which contains $\mathbf{p}$? Which contains $\hat{\mathbf{x}}$? What is the nullspace of A?

Solution: $\mathbf{e}$ is in $\mathbf{N}(A^T)$; $\mathbf{p}$ is in $\mathbf{C}(A)$; $\hat{\mathbf{x}}$ is in $\mathbf{C}(A^T)$; $\mathbf{N}(A) = \{\mathbf{0}\}$ = zero vector only.

Problem 16.6: (4.3 #22.) Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

Solution: The least squares equation is $\begin{bmatrix} 5 & \mathbf{0} \\ \mathbf{0} & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$.
Solution: $C = 1$, $D = -1$. Line $1 - t$. Symmetric t's $\Rightarrow$ diagonal $A^T A$

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