

**DAVID** Hi everyone. I'm Dave.

**SHIROKOFF:**

Now today, I'd like to tackle a problem in orthogonal subspaces. So the problem we'd like to tackle, given a subspace  $S$ , and suppose  $S$  is spanned by two vectors,  $(1, 2, 2, 3)$ , and  $(1, 3, 3, 2)$ . We have a question here which is to find a basis for  $S^\perp$ .  $S^\perp$  is another subspace which is orthogonal to  $S$ . And then secondly, can every vector in  $\mathbb{R}^4$  be uniquely written in terms of  $S$  and  $S^\perp$ . So I'll let you think about this for now, and I'll come back in a minute.

Hi everyone. Welcome back.

OK, so why don't we tackle this problem? OK, so first off, what does it mean for a vector to be an  $S^\perp$ ? Well if I have a vector  $x$ , and  $S^\perp$ , and  $x$  is in  $S^\perp$ , what this means is  $x$  is going to be orthogonal to every vector in  $S$ . Now specifically,  $S$  is spanned by these two vectors. So it's sufficient that  $x$  be perpendicular to the two basis vectors in  $S$ .

So specifically, I can take  $(1, 2, 2, 3)$ , and dot it with  $x$ , and it's going to be 0. So I'm treating  $x$  as a column vector here. In addition,  $x$  must also be orthogonal to  $(1, 3, 2, 2)$ . So any vector  $x$  that's an  $S^\perp$  must be orthogonal to both of these vectors. So what we can do is we can write this as a matrix equation. And we do this by combining these two vectors as rows of the matrix.

So if we step back and take a look at this equation, we see that what we're really asking is to find all  $x$  that are in the null space of this matrix. So how do we find  $x$  in the null space of a matrix? Well what we can do is we can row reduce this matrix and try and find a basis for the null space.

So I'm going to just row reduce this matrix, and notice that by row reduction, we don't actually change the null space of a matrix. So if I'm only interested in the null space, this system is going to be equivalent to, I can keep the top row the same. And then just to simplify our lives, we can take the second row and subtract one

copy of the first row.

Now, if I do that, I obtain  $0, 1, 1, -1$ .

Now, to parameterize the null space, what I'm going to do is I'm going to write  $x$  out as components. So if I write  $x$  with components  $x_1, x_2, x_3$  and  $x_4$ , we see here that this matrix has a rank of 2. Now, we're looking at vectors which live in  $\mathbb{R}^4$ , so we know that the null space is going to have a dimension which is 4 minus 2. So that means there should be two vectors in the null space of this matrix.

To parameterize these two dimensional vectors, what I'm going to do is I'm going to let  $x_4$  equals some constant, and  $x_3$  equal another constant. So specifically, I'm going to let  $x_4$  equal  $b$ , and  $x_3$  equal  $a$ .

Now what we do is we take a look at these two equations, and this bottom equation will say that  $x_2$  is equal to negative  $x_3$  plus  $x_4$ , which is going to equal  $-a, x_4$  plus  $b$ . And then the top equation says that  $x_1$  is equal to  $-2, x_2$  minus  $2x_3$  minus  $3x_4$ , And if I substitute in,  $x_2$  is  $-a$  plus  $b$ .  $x_3$  is  $a$ . And  $x_4$  is  $b$ . So when the dust settles, the  $a$ 's cancel and I'm left with  $-5b$ .

So we can combine everything together and we end up obtaining  $x_1, x_2, x_3, x_4$  equals  $-5b, x_2$  is  $-a$  plus  $b$ .  $x_3$  is  $a$ , and  $x_4$  is  $b$ .

And now what we can do is we can take this vector and we can decompose it into pieces which are  $a$  multiplied by a vector, and  $b$  multiplied by a vector. So you'll note that this is actually  $a$  times  $0, -1, 1, 0$  plus  $b$  times  $-5, 1, 0, 1$ . OK?

So we have successfully achieved a parameterization of the null space of this matrix as some constant  $a$  times a vector  $0, -1, 1, 0$  plus  $b$  times a vector  $-5, 1, 0, 1$ . And now we claim that this is the entire space,  $S$  perp.

So  $S$  perp is going to be spanned by this vector and this vector. Now notice how if I were to take either of these two vectors in  $S$  and dot it with any vector in the null space, by construction it automatically vanishes.

So this concludes part one. Now for part two. Can every vector  $v$  in  $\mathbb{R}^4$  be written

uniquely in terms of  $S$  and  $S^\perp$ ? The answer is yes. So how do we see this?

Well if I have a vector  $v$ , what I can do is I can try and write it as some constant  $c_1$  times the vector  $1, 2, 2, 3$  plus  $c_2$  times the vector  $1, 3, 3, 2$  plus the vector  $c_3, 0, -1, 1, 0$  plus  $c_4, -5, 1, 0, 1$ . OK? So  $c_1$  and  $c_2$  are multiplying the vectors in  $S$ , and  $c_3$  and  $c_4$  are multiplying the vectors in  $S^\perp$ .

So the question is, given any  $v$ , can I find constants  $c_1, c_2, c_3, c_4$ , such that this equation holds? And the answer is yes. Just to see why it's yes, what we can do is we can rewrite this in matrix notation, and there's kind of a handy trick.

What I can do is I can take these columns and write them as columns of the matrix. And this whole expression is actually equivalent to this matrix multiplied by the constant,  $c_1, c_2, c_3, c_4$ . And on the right-hand side, we have the vector  $v$ .

Now, by construction, these vectors are linearly independent. And we know from linear algebra that if we have a matrix with linearly independent columns, the matrix is invertible. What this means is for any  $v$  on the right-hand side, we can invert this matrix and obtain unique coefficients,  $c_1, c_2, c_3, c_4$ . This then gives us a unique decomposition for  $v$  in terms of a piece which is in  $S$ , and the piece which is in  $S^\perp$ . And in general this can be done for any vector space.

Well I'd like to conclude this problem now and I hope you had a good time.