18.06SC Unit 2 Exam Solutions

1 (24 pts.) Suppose q_1, q_2, q_3 are orthonormal vectors in $\mathbb{R}^3$. Find **all possible values** for these 3 by 3 determinants and explain your thinking in 1 sentence each.

(a) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} =$

(b) $\det \begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} =$

(c) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ times $\det \begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix} =$

Solution.

(a) The determinant of any square matrix with orthonormal columns ("orthogonal matrix") is ± 1.

(b) Here are two ways you could do this:

(1) The determinant is *linear in each column*:

$$\begin{aligned}
\det \begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} &= \det \begin{bmatrix} q_1 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} + \det \begin{bmatrix} q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} \\
&= \det \begin{bmatrix} q_1 & q_2 + q_3 & q_3 \end{bmatrix} + \det \begin{bmatrix} q_2 & q_3 & q_3 + q_1 \end{bmatrix} \\
&= \det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} + \det \begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix}
\end{aligned}$$

Both of these determinants are equal (see (c)), so the total determinant is ± 2.

(2) You could also *use row reduction*. Here's what happens:

$$\begin{aligned}
\det\begin{bmatrix} q_1+q_2 & q_2+q_3 & q_3+q_1 \end{bmatrix} &= \det\begin{bmatrix} q_1+q_2 & -q_1+q_3 & q_3+q_1 \end{bmatrix} \\
&= \det\begin{bmatrix} q_1+q_2 & -q_1+q_3 & 2q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_1+q_2 & -q_1+q_3 & q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_1+q_2 & -q_1 & q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_2 & -q_1 & q_3 \end{bmatrix} \\
&= 2\det\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}
\end{aligned}$$

Again, whatever $\det\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ is, this determinant will be twice that, or ± 2.

(c) The second matrix is an *even* permutation of the columns of the first matrix (swap q_1/q_2 then swap q_2/q_3), so it has the *same* determinant as the first matrix. Whether the first matrix has determinant $+1$ or -1, the product will be $+1$.

2 (24 pts.) Suppose we take measurements at the 21 equally spaced times $t = -10, -9, \ldots, 9, 10$. All measurements are $b_i = 0$ except that $b_{11} = 1$ at the middle time $t = 0$.

(a) Using least squares, what are the best $\widehat{C}$ and $\widehat{D}$ to fit those 21 points by a straight line $C + Dt$?

(b) You are projecting the vector b onto what subspace? (*Give a basis.*) Find a nonzero vector perpendicular to that subspace.

Solution.

(a) If the line went exactly through the 21 points, then the 21 equations

$$\begin{bmatrix} 1 & -10 \\ 1 & -9 \\ \vdots & \vdots \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

would be exactly solvable. Since we can't solve this equation $Ax = b$ exactly, we look for a least-squares solution $A^{\mathrm{T}}A\hat{x} = A^{\mathrm{T}}b$.

$$\begin{bmatrix} 21 & 0 \\ 0 & 770 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So the line of best fit is the horizontal line $\widehat{C} = \frac{1}{21}$, $\widehat{D} = 0$.

(b) We are projecting b onto the column space of A above (basis: $\begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} -10 & \ldots & 10 \end{bmatrix}^{\mathrm{T}}$). There are lots of vectors perpendicular to this subspace; one is the error vector $e = b - P_A b = \frac{1}{21} \begin{bmatrix} (\text{ten } -1\text{'s}) & 20 & (\text{ten } -1\text{'s}) \end{bmatrix}^{\mathrm{T}}$.

3 (9 + 12 + 9 pts.) The Gram-Schmidt method produces orthonormal vectors q_1, q_2, q_3 from independent vectors a_1, a_2, a_3 in $\mathbb{R}^5$. Put those vectors into the columns of 5 by 3 matrices Q and A.

(a) Give formulas using Q and A for the projection matrices P_Q and P_A onto the column spaces of Q and A.

(b) ***Is $\boldsymbol{P_Q = P_A}$ and why ? What is $\boldsymbol{P_Q}$ times $\boldsymbol{Q}$? What is* det $\boldsymbol{P_Q}$ *?***

(c) Suppose a_4 is a new vector and a_1, a_2, a_3, a_4 are independent. Which of these (if any) is the new Gram-Schmidt vector q_4? (P_A and P_Q from above)

$$\textbf{1.}\ \frac{P_Q a_4}{\|P_Q a_4\|} \qquad \textbf{2.}\ \frac{a_4 - \frac{a_4^{\mathrm{T}} a_1}{a_1^{\mathrm{T}} a_1} a_1 - \frac{a_4^{\mathrm{T}} a_2}{a_2^{\mathrm{T}} a_2} a_2 - \frac{a_4^{\mathrm{T}} a_3}{a_3^{\mathrm{T}} a_3} a_3}{\|\text{ norm of that vector }\|} \qquad \textbf{3.}\ \frac{a_4 - P_A a_4}{\|a_4 - P_A a_4\|}$$

Solution.

(a) $P_A = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ and $P_Q = Q(Q^{\mathrm{T}}Q)^{-1}Q^{\mathrm{T}} = QQ^{\mathrm{T}}$.

(b) $P_A = P_Q$ because both projections project onto the same subspace. *(Some people did this the hard way, by substituting $A = QR$ into the projection formula and simplifying. That also works.)* The determinant is zero, because P_Q is singular (like all non-identity projections): all vectors orthogonal to the column space of Q are projected to 0.

(c) Answer: choice 3. (Choice 2 is tempting, and would be correct if the a_i were replaced by the q_i. But the a_i are not orthogonal!)

4 (22 pts.) Suppose a 4 by 4 matrix has the same entry $\times$ throughout its first row and column. The other 9 numbers could be anything like $1, 5, 7, 2, 3, 99, \pi, e, 4$.

$$A = \begin{bmatrix} \times & \times \quad \times \quad \times \\ \times & \text{any numbers} \\ \times & \text{any numbers} \\ \times & \text{any numbers} \end{bmatrix}$$

(a) The determinant of A is a polynomial in $\times$. What is the largest possible degree of that polynomial? **Explain your answer**.

(b) If those 9 numbers give the identity matrix I, what is $\det A$? Which values of $\times$ give $\det A = 0$?

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & 1 & 0 & 0 \\ \times & 0 & 1 & 0 \\ \times & 0 & 0 & 1 \end{bmatrix}$$

Solution.

(a) Every term in the big formula for $\det(A)$ takes one entry from each row and column, so we can choose at most two $\times$'s and the determinant has degree 2.

(b) You can find this by cofactor expansion; here's another way:

$$\det(A) = \times \det \begin{bmatrix} 1 & \times & \times & \times \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \times \det \begin{bmatrix} 1-3\times & \times & \times & \times \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \times(1-3\times) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= \times(1 - 3\times)$.

This is zero when $\times = 0$ or $\times = \frac{1}{3}$.