

BEN HARRIS: Hi, and welcome back. Today we're going to do a problem about the four fundamental subspaces. So here we have a matrix  $B$ .  $B$  is written as the product to a lower triangular matrix and an upper triangular matrix. And we're going to find a basis for, and compute the dimension of, each of the four fundamental subspaces of  $B$ .

I'll give you a minute to try that on your own, to hit pause, and then I'll be right back in just a minute and we can do it together.

OK. We're back. Now, the first thing to notice is that this is an LU decomposition of  $B$ . So we have  $L$  here and  $U$  here. Now let's go one space at a time. Let's start with the column space. And first, let's just say what the dimension of the column space is. OK, so let's look at our  $U$  matrix. How many pivots do we have? We have two pivots, so the column space has dimension 2. This is the number of pivots.

Good. Now how do we find a basis for the column space? How do we find that basis? Well, in lecture Professor Strang had started with a matrix  $B$ . He did elimination on it, and then he took the pivot columns in the original matrix. And that's great. That works. You can also take the pivot columns in the  $L$  matrix. You can see by multiplying this out that it will amount to essentially the same thing.

So a basis for this column space, I can just take these two pivot columns of my  $L$  matrix, 1, 2, -1 and 0, 1, 0. Good. OK. So there's the dimension of and the basis for the column space of  $B$ .

Next, let's do the null space together. OK. What's the dimension of the null space? Well, the dimension of the null space is always the number of columns minus the number of pivots. Right? It's the number of free variables. So here, that's just one. Good.

And how do we find this one vector in the null space? Well, what we do is we can just plug in 1 for our free variable, and we can back solve to get the other two. So this equation tells me that my second number is -1, and this equation tells me that that third variable  $-3/5$ , if I can fit it in here. That's a  $-3/5$ , if it's difficult to see that on the tape.

Now let's move on. Next is the row space. So how do we find the dimension of the row space? I'm going to write row space as column space of  $B$  transpose. How do we find that? Well, remember that one of our big facts in this class is that the dimension of the row space is the same as the dimension of the column space. It's just the number of pivots. So that's good. It's 2. And how do we find a basis for the row space? There are a couple ways of thinking about this.

One way to think about it is we got this upper triangular matrix from  $B$  by doing elimination. And elimination doesn't change the row space. So I can just use the two pivot rows of the matrix  $u$ . Basis for my row space here is I just put these two pivot rows together, and I get a basis for this row space.

The last one is always the toughest and the trickiest. We have to do the left null space or the null space of  $B$  transpose. First, let's compute its dimension. What's the dimension of this left null space? Well, there's a similar formula to the one we used when we were computing the dimension of the null space. It's just the number of rows minus the number of pivots. So there are three rows. Our matrix is 3 by 3. And there're other two pivots. So this is just one dimensional, again.

We need to compute, now, this left null space. Let me go back to our original matrix. The way to do this is to take  $B$  equals  $LU$ , and invert  $L$ , and get  $EB$  equals  $U$ . So we need to move  $L$  over to the left-hand side. If we do that-- I'm just going to write that down here. So what's the inverse of the  $L$  matrix? We just get 1, -2, 1, 0, 1, 1, times  $B$ , is our  $U$  matrix, this upper-triangular matrix. Now that I moved  $L$  over to the other side, I can read off the vectors in my left null space.

Now, I'm looking at not my pivot variables but my free variables, because it's some sort of null space. but I want to look at this  $E$  matrix. So the third row of this  $E$  matrix, the third row corresponds to the three row here and when I multiply this by  $B$ , I just get zeros, so this is in the left null space. A basis for this left null space is-- see if I can fit it here-- just this 1, 0, 1. Good.

So we've found the dimension of and basis for all of the four fundamental subspaces. Before I move on, I just want to recall which of the  $L$  matrix or the  $U$  matrix we used for each of these subspaces. So for the column space, we used the pivot columns of the  $L$  matrix. For the null space, we looked at the  $U$  matrix. For the row space we also looked at the  $U$  matrix. And for the left null space, we needed to invert the  $L$  matrix and look at the three row.

We're done with the problem. But the last thing that's useful is to draw a picture, which I have right here. I know in lecture Professor Strang has drawn you some sort of cartoon pictures of what these subspaces look like. But here I want to try to actually draw them in a special case. So if you can read my drawing here, what do we have? We have the row space here, and the null space here. Right? And so  $B$  maps this picture into this picture.

The null space here-- all the scalar multiples of this vector-- all go to 0, because they're in the null space. That's exactly what  $B$  takes to 0.  $B$  takes everything else, including the row space, into this column space. And what is  $B$  transpose do? Well  $B$  transpose kills this left null space, kills this vector, and it take everything else into the row space, into the column space of  $B$  transpose. OK.

Thanks for doing this exercise together. I hope this picture is helpful.

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