

## Exercises on solving $Ax = \mathbf{b}$ and row reduced form $R$

**Problem 8.1:** (3.4 #13.(a,b,d) *Introduction to Linear Algebra: Strang*) Explain why these are all false:

- a) The complete solution is any linear combination of  $\mathbf{x}_p$  and  $\mathbf{x}_n$ .
- b) The system  $Ax = \mathbf{b}$  has at most one particular solution.
- c) If  $A$  is invertible there is no solution  $\mathbf{x}_n$  in the nullspace.

**Solution:**

- a) The coefficient of  $\mathbf{x}_p$  must be one.
- b) If  $\mathbf{x}_n \in \mathbf{N}(A)$  is in the nullspace of  $A$  and  $\mathbf{x}_p$  is one particular solution, then  $\mathbf{x}_p + \mathbf{x}_n$  is also a particular solution.
- c) There's always  $\mathbf{x}_n = \mathbf{0}$ .

**Problem 8.2:** (3.4 #28.) Let

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Use Gauss-Jordan elimination to reduce the matrices  $[U \ 0]$  and  $[U \ \mathbf{c}]$  to  $[R \ 0]$  and  $[R \ \mathbf{d}]$ . Solve  $R\mathbf{x} = \mathbf{0}$  and  $R\mathbf{x} = \mathbf{d}$ .

Check your work by plugging your values into the equations  $U\mathbf{x} = \mathbf{0}$  and  $U\mathbf{x} = \mathbf{c}$ .

**Solution:** First we transform  $[U \ 0]$  into  $[R \ 0]$ :

$$[U \ 0] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [R \ 0].$$

We now solve  $R\mathbf{x} = \mathbf{0}$  via back substitution:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + 2x_2 = 0 \\ x_3 = 0 \end{bmatrix} \longrightarrow \mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix},$$

where we used the free variable  $x_2 = -1$ . ( $c\mathbf{x}$  is a solution for all  $c$ .)

We check that this is a correct solution by plugging it into  $U\mathbf{x} = \mathbf{0}$ :

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark$$

Next, we transform  $[U \ \mathbf{c}]$  into  $[R \ \mathbf{d}]$ :

$$[U \ \mathbf{c}] = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = [R \ \mathbf{d}].$$

We now solve  $R\mathbf{x} = \mathbf{d}$  via back substitution:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + 2x_2 = -1 \\ x_3 = 2 \end{bmatrix} \longrightarrow \mathbf{x} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix},$$

where we used the free variable  $x_2 = 1$ .

Finally, we check that this is the correct solution by plugging it into the equation  $U\mathbf{x} = \mathbf{c}$ :

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \checkmark$$

**Problem 8.3:** (3.4 #36.) Suppose  $A\mathbf{x} = \mathbf{b}$  and  $C\mathbf{x} = \mathbf{b}$  have the same (complete) solutions for every  $\mathbf{b}$ . Is it true that  $A = C$ ?

**Solution:** **Yes.** In order to check that  $A = C$  as matrices, it is enough to check that  $A\mathbf{y} = C\mathbf{y}$  for all vectors  $\mathbf{y}$  of the correct size (or just for the standard basis vectors, since multiplication by them “picks out the columns”). So let  $\mathbf{y}$  be any vector of the correct size, and set  $\mathbf{b} = A\mathbf{y}$ . Then  $\mathbf{y}$  is certainly a solution to  $A\mathbf{x} = \mathbf{b}$ , and so by our hypothesis must also be a solution to  $C\mathbf{x} = \mathbf{b}$ ; in other words,  $C\mathbf{y} = \mathbf{b} = A\mathbf{y}$ .

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