Exercises on matrix spaces; rank 1; small world graphs

Problem 11.1: [Optional] (3.5 #41. *Introduction to Linear Algebra:* Strang) Write the 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1 + \cdots + c_5P_5 = 0$ and check entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of three by three matrices with row and column sums all equal.

Solution: The other five permutation matrices are:

$$P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix},$$

$$P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \text{ and } P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

Since $P_{21} + P_{31} + P_{32}$ is the all ones matrix and $P_{32}P_{21} + P_{21}P_{32}$ is the matrix with zeros on the diagonal and ones elsewhere,

$$I = P_{21} + P_{31} + P_{32} - P_{32}P_{21} - P_{21}P_{32}.$$

For the second part, setting $c_1P_1 + \cdots + c_5P_5$ equal to zero gives:

$$\begin{bmatrix} c_3 & c_1 + c_4 & c_2 + c_5 \\ c_1 + c_5 & c_2 & c_3 + c_4 \\ c_2 + c_4 & c_3 + c_5 & c_1 \end{bmatrix} = 0.$$

So $c_1 = c_2 = c_3 = 0$ along the diagonal, and $c_4 = c_5 = 0$ from the off-diagonal entries.

Problem 11.2: (3.6 #31.) **M** is the space of three by three matrices. Multiply each matrix X in **M** by:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Notice that $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

a) Which matrices X lead to $AX = 0$?

b) Which matrices have the form AX for some matrix X?

c) Part (a) finds the "nullspace" of the operation AX and part (b) finds the "column space." What are the dimensions of those two subspaces of $\mathbf{M}$? Why do the dimensions add to $(n-r)+r=9$?

Solution:

a) We can use row reduction or some other method to see that the rows of A are dependent and that A has rank 2. Its nullspace has the basis:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$AX = 0$ precisely when the columns of X are in the nullspace of A, i.e. when they are multiples of the basis of $N(A)$. Therefore, if $AX = 0$ then X must have the form:

$$X = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}.$$

b) On the other hand, the columns of any matrix of the form AX are linear combinations of the columns of A. That is, they are vectors whose components all sum to 0, so a matrix has the form AX if and only if all of its columns individually sum to 0:

$$AX = B \text{ if and only if } B = \begin{bmatrix} a & b & c \\ d & e & f \\ -a-d & -b-e & -c-f \end{bmatrix}.$$

c) The dimension of the "nullspace" is 3, while the dimension of the "column space" is 6. These add up to 9, which is the dimension of the space of "inputs" $\mathbf{M}$.