

## Exercises on column space and nullspace

**Problem 6.1:** (3.1 #30. *Introduction to Linear Algebra: Strang*) Suppose  $\mathbf{S}$  and  $\mathbf{T}$  are two subspaces of a vector space  $\mathbf{V}$ .

- a) **Definition:** The sum  $\mathbf{S} + \mathbf{T}$  contains all sums  $\mathbf{s} + \mathbf{t}$  of a vector  $\mathbf{s}$  in  $\mathbf{S}$  and a vector  $\mathbf{t}$  in  $\mathbf{T}$ . Show that  $\mathbf{S} + \mathbf{T}$  satisfies the requirements (addition and scalar multiplication) for a vector space.
- b) If  $\mathbf{S}$  and  $\mathbf{T}$  are lines in  $\mathbf{R}^m$ , what is the difference between  $\mathbf{S} + \mathbf{T}$  and  $\mathbf{S} \cup \mathbf{T}$ ? That union contains all vectors from  $\mathbf{S}$  and  $\mathbf{T}$  or both. Explain this statement: *The span of  $\mathbf{S} \cup \mathbf{T}$  is  $\mathbf{S} + \mathbf{T}$ .*

**Solution:**

- a) Let  $\mathbf{s}, \mathbf{s}'$  be vectors in  $\mathbf{S}$ , let  $\mathbf{t}, \mathbf{t}'$  be vectors in  $\mathbf{T}$ , and let  $c$  be a scalar. Then

$$(\mathbf{s} + \mathbf{t}) + (\mathbf{s}' + \mathbf{t}') = (\mathbf{s} + \mathbf{s}') + (\mathbf{t} + \mathbf{t}') \text{ and } c(\mathbf{s} + \mathbf{t}) = c\mathbf{s} + c\mathbf{t}.$$

Thus  $\mathbf{S} + \mathbf{T}$  is closed under addition and scalar multiplication; in other words, it satisfies the two requirements for a vector space.

- b) If  $\mathbf{S}$  and  $\mathbf{T}$  are distinct lines, then  $\mathbf{S} + \mathbf{T}$  is a plane, whereas  $\mathbf{S} \cup \mathbf{T}$  is only the two lines. The span of  $\mathbf{S} \cup \mathbf{T}$  is the set of all combinations of vectors in this union of two lines. In particular, it contains all sums  $\mathbf{s} + \mathbf{t}$  of a vector  $\mathbf{s}$  in  $\mathbf{S}$  and a vector  $\mathbf{t}$  in  $\mathbf{T}$ , and these sums form  $\mathbf{S} + \mathbf{T}$ .

Since  $\mathbf{S} + \mathbf{T}$  contains both  $\mathbf{S}$  and  $\mathbf{T}$ , it contains  $\mathbf{S} \cup \mathbf{T}$ . Further,  $\mathbf{S} + \mathbf{T}$  is a vector space. So it contains all combinations of vectors in itself; in particular, it contains the span of  $\mathbf{S} \cup \mathbf{T}$ . Thus the span of  $\mathbf{S} \cup \mathbf{T}$  is  $\mathbf{S} + \mathbf{T}$ .

**Problem 6.2:** (3.2 #18.) The plane  $x - 3y - z = 12$  is parallel to the plane  $x - 3y - x = 0$ . One particular point on this plane is  $(12, 0, 0)$ . All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

**Solution:** The equation  $x = 12 + 3y + z$  says it all:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \left( = \begin{bmatrix} 12 + 3y + z \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

**Problem 6.3:** (3.2 #36.) How is the nullspace  $\mathbf{N}(C)$  related to the spaces  $\mathbf{N}(A)$  and  $\mathbf{N}(B)$ , if  $C = \begin{bmatrix} A \\ B \end{bmatrix}$ ?

**Solution:**  $\mathbf{N}(C) = \mathbf{N}(A) \cap \mathbf{N}(B)$  contains all vectors that are in both nullspaces:

$$C\mathbf{x} = \begin{bmatrix} A\mathbf{x} \\ B\mathbf{x} \end{bmatrix} = \mathbf{0}$$

if and only if  $A\mathbf{x} = \mathbf{0}$  and  $B\mathbf{x} = \mathbf{0}$ .

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