SIMPLE AND POSITIVE ROOTS

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Let V be a Euclidean space, i.e. a real finite dimensional linear space with a symmetric positive definite inner product $\langle , \rangle$.

We recall that a root system in V is a finite set Δ of nonzero elements of V such that

(1) Δ spans V;
(2) for all $\alpha \in \Delta$, the reflections
$$s_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$
map the set Δ to itself;
(3) the number $\frac{2\langle \beta,\alpha \rangle}{\langle \alpha,\alpha \rangle}$ is an integer for any $\alpha, \beta \in \Delta$.

A root is an element of Δ.

Here are two examples of root systems in $\mathbb{R}^2$:

Example 1. *The root system of the type $A_1 \oplus A_1$ consists of the four vectors $\{\pm e_1, \pm e_2\}$ where $\{e_1, e_2\}$ is an orthonormal basis in $\mathbb{R}^2$.*

We note that condition (1) is satisfied because $\{e_1, e_2\}$ spans $\mathbb{R}^2$. Also, since $\langle \pm e_1, \pm e_2 \rangle = 0$ it follows that $s_{e_i}(e_j) = s_{-e_i}(e_j) = e_j$ and $s_{e_i}(-e_j) = s_{-e_i}(-e_j) = -e_j$ for $i \neq j$. Similarily, $\langle e_i, e_i \rangle = 1$ and $\langle e_i, -e_i \rangle = -1$ give that $s_{e_i}(e_i) = s_{-e_i}(e_i) = -e_i$, and $s_{e_i}(-e_i) = s_{-e_i}(-e_i) = e_i$,. Thus, conditions (2) and (3) are also satisfied. For a sketch of $A_1 \oplus A_1$, see Figure 1 on page 6.

Example 2. *The root system of the type A_2 consists of the six vectors $\{e_i - e_j\}_{i \neq j}$ in the plane orthogonal to the line $e_1 + e_2 + e_3$ where $\{e_1, e_2, e_3\}$ is an orthonormal basis in $\mathbb{R}^3$. These roots can be rewritten in a standard orthonormal basis of the plane for a more illustrative description in $\mathbb{R}^2$.*

We choose, as our standard orthonormal basis for the plane, vectors $\{i, j\}$ such that $i = e_2 - e_1$ and for $d = (e_3 - e_1) + (e_3 - e_2)$, $j = |i|/|d| \cdot d = (2e_3 - e_2 - e_1)/\sqrt{3}$. It is easy to verify that $\langle i, j \rangle = 0$. Further, we choose as our unit length $|i| = |j| = \sqrt{2}$. Then, all the roots $\alpha \in \Delta$ can be represented as $\alpha = cos(n\pi/3) \cdot i + sin(n\pi/3) \cdot j$ for $n = 0, 1, 2, 3, 4, 5$. That is, all the roots lie on a unit circle and the angle between any two such roots is an integer multiple of $\pi/3$. E.g. for $n = 1$ we obtain $\alpha = cos(\pi/3) \cdot i + sin(\pi/3) \cdot j =$

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$\frac{1}{2} \cdot i + \frac{\sqrt{3}}{2} \cdot j = \frac{1}{2} \cdot (e_2 - e_1) + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{3}}(2e_3 - e_2 - e_1) = e_3 - e_1 \in \Delta$. Other cases can easily be verified. For a sketch of A_2, see Figure 2 on page 6.

Since for any $\alpha \in \Delta$, $-\alpha$ is also in Δ, (see [1], Thm.8(1)), the number of elements in Δ is always greater than the dimension of V. The example of type A_2 above shows that even a subset of mutually noncollinear vectors in Δ might be too big to be linearly independent. In the present paper we would like to define a subset of Δ small enough to be a basis in V, yet large enough to contain the essential information about the geometric properties of Δ. Here is a formal definition.

Definition 3. *A subset* Π *in* Δ *is a set of simple roots (a simple root system) in* Δ *if*

1. Π *is a basis in* V*;*
2. *Each root* $\beta \in \Delta$ *can be written as a linear combination of the elements of* Π *with integer coefficients of the same sign, i.e.*
$$\beta = \sum_{\alpha \in \Pi} m_\alpha \alpha$$
with all $m_\alpha \geq 0$ *or all* $m_\alpha \leq 0$.

The root β *is positive if the coefficients are nonnegative, and negative otherwise. The set of all positive roots (positive root system) associated to* Π *will be denoted* Δ^+.

We will now construct a set Π_t associated to an element $t \in V$ and a root system Δ, and show that it satisfies the definition of a simple root system in Δ.

Let Δ be a root system in V, and let $t \in V$ be a vector such that $\langle t, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$. Set
$$\Delta_t^+ = \{\alpha \in \Delta : \langle t, \alpha \rangle > 0\}.$$
Let $\Delta_t^- = \{-\alpha, \alpha \in \Delta_t^+\}$.

Remark. *It is always possible to find* $t \in V$ *such that* $\langle t, \alpha \rangle \neq 0$ *for any* $\alpha \in \Delta$.

We note that Δ has a finite number of elements and thus there is only a finite number of hyperplanes H_α such that for any $t \in H_\alpha, \langle t, \alpha \rangle = 0$. Furthermore, since $\dim H_\alpha = \dim V - 1$ it is clear that $\bigcup_{\alpha \in \Delta} H_\alpha$ cannot span V and thus we can always find $t \in V$ such that $\langle t, \alpha \rangle \neq 0$ for any $\alpha \in \Delta$.

Remark. $\Delta = \Delta_t^+ \cup \Delta_t^-$.

We know that $\langle t, \alpha \rangle \neq 0$ for any $\alpha \in \Delta$. Also, for $\alpha \in \Delta$ necessarily $-\alpha \in \Delta$. Since, $\langle t, -\alpha \rangle = -\langle t, \alpha \rangle$ it must be that either $\langle t, \alpha \rangle > 0$ or $\langle t, -\alpha \rangle > 0$, and $\alpha \in \Delta_t^+$ or $\alpha \in \Delta_t^-$ respectively. Thus, $\Delta_t^+ \cup \Delta_t^- = \Delta$.

Definition 4. *An element* $\alpha \in \Delta_t^+$ *is decomposable if there exist* $\beta, \gamma \in \Delta_t^+$ *such that* $\alpha = \beta + \gamma$. *Otherwise* $\alpha \in \Delta_t^+$ *is indecomposable.*

Let $\Pi_t \subset \Delta_t^+$ be the set of all indecomposable elements in Δ_t^+.

The next three Lemmas prove the properties of Δ_t^+ and Π_t.

Lemma 5. *Any element in Δ_t^+ can be written as a linear combination of elements in Π_t with nonnegative integer coefficients.*

Proof. By contradiction. Suppose γ is an element of Δ_t^+ for which the lemma is false. Since Δ_t^+ is a finite set we can choose such a γ for which $\langle t, \gamma \rangle > 0$ is minimal. Since $\gamma \in \Delta_t^+$ but $\gamma \notin \Pi_t$, γ must be decomposable. Hence, $\gamma = \alpha + \beta$ and $\langle t, \gamma \rangle = \langle t, \alpha + \beta \rangle = \langle t, \alpha \rangle + \langle t, \beta \rangle$. Furthermore, since $\alpha, \beta \in \Delta_t^+$, $\langle t, \alpha \rangle > 0$ and $\langle t, \beta \rangle > 0$ it must be that $\langle t, \gamma \rangle > \langle t, \alpha \rangle$ and $\langle t, \gamma \rangle > \langle t, \beta \rangle$. By the minimality of $\langle t, \gamma \rangle$ this Lemma must then hold for α and β. However, then it must also hold for $\gamma = \alpha + \beta$, which is a contradiction. Thus, such a γ cannot exist and the lemma holds. □

Lemma 6. *If $\alpha, \beta \in \Pi_t$, $\alpha \neq \beta$, then $\langle \alpha, \beta \rangle \leq 0$.*

Proof. By contradiction. Suppose that $\langle \alpha, \beta \rangle > 0$. Then by Theorem 9(1) in [1] $\alpha - \beta \in \Delta$ or $\alpha - \beta = 0$. We do not consider the latter case since then $\alpha = \beta$. However, considering $\alpha - \beta = \gamma \in \Delta$ for $\alpha, \beta \in \Pi_t$. Then, $\gamma \in \Delta_t^+$ or $\gamma \in \Delta_t^-$. In the first case we find that $\alpha = \gamma + \beta$. However, α is indecomposable in Δ_t^+ and we have a contradiction. In the latter case, since then $-\gamma \in \Delta_t^+$, we find that $\beta = -\gamma + \alpha$. However, β is also indecomposable in Δ_t^+ and again we have a contradiction. Hence, the Lemma holds. □

Remark. *If we consider a euclidean space with a standard dot-product for $\langle \alpha, \beta \rangle = |\alpha||\beta| cos(\phi) \leq 0$ it is clear by previous lemma that the smallest angle ϕ between the vectors satisfies $\pi/2 \leq \phi \leq \pi$.*

Lemma 7. *Let A be a subset of V such that*

(1) $\langle t, \alpha \rangle > 0$ *for all* $\alpha \in A$;

(2) $\langle \alpha, \beta \rangle \leq 0$ *for all* $\alpha, \beta \in A$.

Then the elements of A are linearly independent.

Proof. By contradiction. Suppose that the elements of A are linearly dependent. Then for $\alpha_i \in A$ we can form $\sum c_i \alpha_i = 0$ such that not all $c_i = 0$. Since some $c_i > 0$ and also some $c_i < 0$, we split the linear combination into two sums with all positive coefficients and obtain $\sum m_\beta \beta - \sum n_\gamma \gamma = 0$ with $\beta, \gamma \in A$ and all $m_\beta, n_\gamma > 0$. We then denote $\lambda = \sum m_\beta \beta = \sum n_\gamma \gamma$ and consider $\langle \lambda, \lambda \rangle \geq 0$ (by definite positive property of inner product). Then also $\langle \lambda, \lambda \rangle = \langle \sum m_\beta \beta, \sum n_\gamma \gamma \rangle = \sum m_\beta \sum n_\gamma \langle \beta, \gamma \rangle$. However, since $\langle \beta, \gamma \rangle \leq 0$ by initial assumption and all $m_\beta, n_\gamma > 0$ we obtain that $\langle \lambda, \lambda \rangle \leq 0$. Thus $\langle \lambda, \lambda \rangle = 0$ and necessarily $\lambda = \vec{0}$. If we then consider $\langle \lambda, t \rangle = \langle \sum m_\beta \beta, t \rangle = \sum m_\beta \langle \beta, t \rangle = 0$ and note that by initial assumption $\langle t, \alpha \rangle > 0$, it must be that all $m_\beta = 0$. Similarily for m_γ. Hence, all $c_i = 0$ and we have a contradiction. Thus, the elements in A are linearly independent. □

Now we are ready to prove the existence of a simple root set in any abstract root system.

Theorem 8. *For any $t \in V$ such that $\langle t, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$, the set Π_t constructed above is a set of simple roots, and Δ_t^+ the associated set of positive roots.*

Proof. We know by Lemma 5 that every element in Δ_t^+ can be written as a linear combination of elements in Π_t with non-negative coefficients. Accordingly, all elements in Δ_t^- can be written with non-positive coefficients. Since $\Delta_t^+ \cup \Delta_t^- = \Delta$, condition (2) is satisfied. Furthermore, for any $\alpha, \beta \in \Pi_t$ we have $\langle \alpha, \beta \rangle \leq 0$ by Lemma 6. Since by construction $\langle t, \alpha \rangle, \langle t, \beta \rangle > 0$ we find by Lemma 7 that all elements in Π_t are linearly independent. Noting that every element of Δ can be written as a linear combination of elements of Π_t and since, by definition, Δ spans V, we conclude that Π_t is a linearly independent set that spans V and thus it is a basis, satisfying condition (1). □

The converse statement is also true:

Theorem 9. *Let Π be a set of simple roots in Δ, and suppose that $t \in V$ is such that $\langle t, \alpha \rangle > 0$ for all $\alpha \in \Pi$. Then $\Pi = \Pi_t$, and the associated set of positive roots $\Delta^+ = \Delta_t^+$.*

Proof. Given t as above, we define Δ_t^+ as before. It is easy to see that $\Delta^+ \subset \Delta_t^+$ since Δ^+ is positive with regards to Π (i.e. any $\alpha \in \Delta^+$ is a linear combination of elements of Π with non-negative coefficients) and Π is positive with regards to t (i.e. $\langle t, \alpha \rangle \geq 0$ for all $\alpha \in \Pi$). Also, similarily $\Delta^- \subset \Delta_t^-$. However, $\Delta = \Delta^+ \cup \Delta^- = \Delta_t^+ \cup \Delta_t^-$. Therefore, the number of elements in Δ^+ is equal to the number of elements in Δ_t^+ and they coincide. Furthermore, Π is a set of simple roots, i.e. it is a basis in V and its elements are indecomposable. Therefore, $\Pi \subset \Pi_t$ where Π_t is defined as all the indecomposable elements in Δ_t^+. However, Π_t is also a basis and therefore the number of elements in Π and Π_t coincide and thus $\Pi = \Pi_t$. □

Example 10. *Let V be the n-dimensional subspace of $\mathbb{R}^{n+1}$ $(n \geq 1)$ orthogonal to the line $e_1 + e_2 + \ldots + e_{n+1}$, where $\{e_i\}_{i=1}^{n+1}$ is an orthonormal basis in $\mathbb{R}^{n+1}$. The root system Δ of the type A_n in V consists of all vectors $\{e_i - e_j\}_{i \neq j}$. Furthermore, $\Pi = \{e_1 - e_2, e_2 - e_3, \ldots, e_n - e_{n+1}\}$ is a set of simple roots, and $\Delta^+ = \{e_i - e_j\}_{i<j}$ - the associated set of positive roots in Δ.*

In order to show that all elements in Δ^+ can be represented by elements of Π with non-negative coefficients we consider $(e_i - e_j)_{i<j} = (e_i - e_{i+1}) + \cdots + (e_{j-1} - e_j)$. Also, for any $\beta \in \Delta^- = \{e_i - e_j\}_{j<i}$ we can simply take the corresponding $\alpha \in \Delta^+$ s.t. $-\alpha = -(e_i - e_j)_{i<j} = (e_j - e_i)_{i<j} = \beta$ and all the coefficients will be non-positive. Since $\{e_i - e_j\}_{i<j} \cup \{e_i - e_j\}_{j<i} = \{e_i - e_j\}_{i \neq j}$ condition (2) is satisfied.

We note that, by above, any element of Δ^+, and thus Δ^-, can be represented as a linear combination of elements of Π. Also, $\Delta = \Delta^+ \cup \Delta^-$ and, by definition, Δ spans V. It follows that Π spans V. We then have n vectors that span an n-dimensional space. They must be linearly independent and form a basis. Thus, condition (1) is satisfied. For a sketch of case $n = 2$, see Figure 2 on page 6.

Example 11. *The root system Δ of the type C_n in $V = \mathbb{R}^n$ ($n \geq 2$) consists of all vectors $\{\pm e_i \pm e_j\}_{i \neq j} \cup \{\pm 2e_i\}$, where $\{e_i\}_{i=1}^n$ is an orthonormal basis in $\mathbb{R}^n$. Furthermore, $\Pi = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, 2e_n\}$ is a set of simple roots, and $\Delta^+ = \{e_i \pm e_j\}_{i<j} \cup \{2e_i\}$ - the associated set of positive roots in Δ.*

In order to show that all elements in Δ^+ can be represented as a linear combination of elements of Π with non-negative coefficients we recall that $(e_i - e_j)_{i<j} = (e_i - e_{i+1}) + \cdots + (e_{j-1} - e_j)$. Also, $2e_j = 2(e_j - e_{j+1}) + \cdots + 2(e_{n-1} - e_n) + 2e_n$. Finally, $(e_i + e_j)_{i<j} = (e_i - e_j)_{i<j} + 2e_j$ using the two previous formulas. Multiplying these formulas by -1 we obtain the elements of Δ^- with all non-positive coefficients. Noting that $\Delta = \Delta^+ \cup \Delta^-$ we see that condition (2) is satisfied. Condition (1) for simple root systems is satisfied by the same argument as in the previous example. For a sketch of C_2, see Figure 3 on page 6.

Example 12. *We let $V = \mathbb{R}^2$ and recall from* [1], *that for any two roots $\alpha, \beta \in \Delta$, $n(\alpha, \beta) \cdot n(\beta, \alpha) = 4\cos^2(\phi)$, where $n(\alpha, \beta) = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}$, and ϕ is the angle between α and β. Using Lemma 6 we can find all the angles between simple roots in $\mathbb{R}^2$ and also their relative lengths. Furthermore, in accordance with Theorem 9, we can define the set of all elements $t \in V$ such that $\Pi_t = \Pi$ for a given Π. This set is the dominant Weyl chamber $C(\Delta, \Pi)$.*

Let us assume that the root system Δ is reduced, that is for any $\alpha \in \Delta$, $2\alpha \notin \Delta$. We have the natural constraint that $n(\alpha, \beta) \cdot n(\beta, \alpha) = 4\cos^2(\phi) \leq 4$. Also, by Lemma 6 for any $\alpha, \beta \in \Pi$, $\langle \alpha, \beta \rangle = |\alpha||\beta|cos(\phi) \leq 0$ and necessarily $90 \leq \phi \leq 180$. Then, by [1] we know that for such α, β, $n(\alpha, \beta) = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \frac{2|\beta||\alpha|cos(\phi)}{|\alpha|^2} = 2\frac{|\beta|}{|\alpha|}cos(\phi) = 0, -1, -2, -3 \text{ or} -4$. By our formula, we obtain $n(\alpha, \beta) \cdot n(\beta, \alpha) = 4\cos^2(\phi) = 0, 1, 2$ or 3, and consider the possible combinations that satisfy this relation. We exclude 4, since in that case α and β are collinear and such a Π could not form a basis, as required. To further illustrate these relations, we can write $\frac{|\alpha|}{|\beta|} = \frac{2cos(\phi)}{n(\alpha,\beta)} = \frac{-\sqrt{4cos^2(\phi)}}{n(\alpha,\beta)}$. and, $\phi = 180 - cos^{-1}\left(\frac{1}{2}\sqrt{n(\alpha, \beta) \cdot n(\beta, \alpha)}\right)$. The results are tabulated in Table 1, page 6.

Figures 1, 2 and 3 sketch the relevant rootsystems and illustrate the dominant Weyl Chambers for all the above mentioned cases. In each case a set of simple roots is denoted by thick arrows. The associated regions for Weyl Chambers are obtained from constraint $C(\Delta, \Pi) = \{t \in V : \langle t, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Pi\}$.

References

[1] 18.06CI - Final Project #2, *Abstract root systems*, MIT, 2004.

$n_{\alpha,\beta}$	$n_{\beta,\alpha}$	$4\,cos^2(\phi)$	ϕ	$\frac{\lvert\alpha\rvert}{\lvert\beta\rvert}$	In type
0	0	0	90	–	$A_1 \oplus A_1$
-1	-1	1	120	1	A_2
-1	-2	2	135	$\sqrt{2}$	C_2
-1	-3	3	150	$\sqrt{3}$	G_2

Table 1:
Possible relations between simple roots

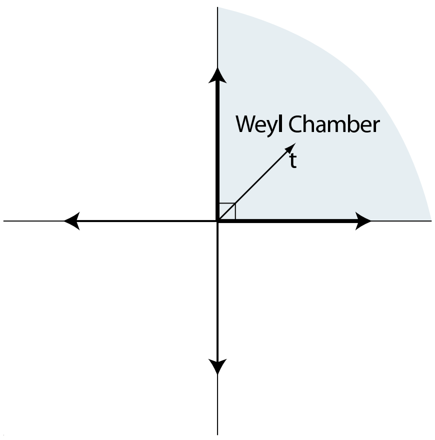

Figure 1:
Root system: $A_1 \oplus A_1$

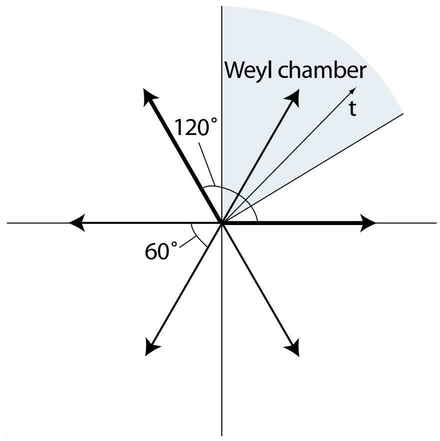

Figure 2:
Root system: A_2

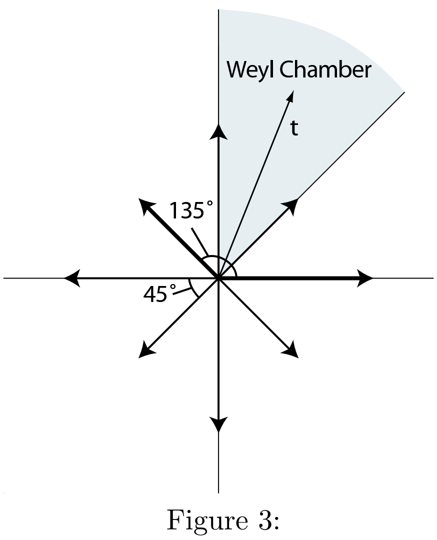

Figure 3:
Root system: C_2

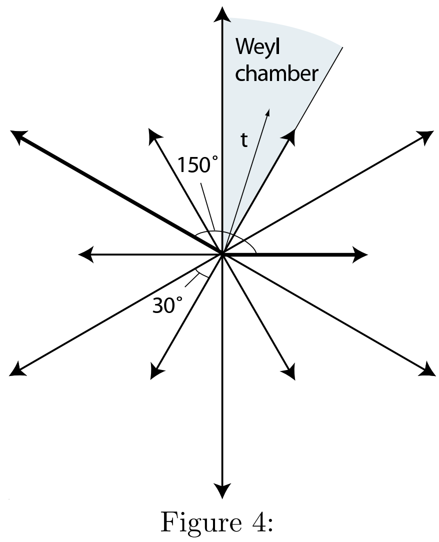

Figure 4:
Root system: G_2