

## REFLECTIONS IN A EUCLIDEAN SPACE

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Let  $V$  be a finite dimensional real linear space.

**Definition 1.** A function  $\langle, \rangle : V \times V \rightarrow \mathbb{R}$  is a bilinear form in  $V$  if for all  $x_1, x_2, x, y_1, y_2, y \in V$  and all  $k \in \mathbb{R}$ ,

$$\begin{aligned}\langle x_1 + kx_2, y \rangle &= \langle x_1, y \rangle + k\langle x_2, y \rangle, \text{ and} \\ \langle x, y_1 + ky_2 \rangle &= \langle x, y_1 \rangle + k\langle x, y_2 \rangle.\end{aligned}$$

**Definition 2.** A bilinear form  $\langle, \rangle$  in  $V$  is symmetric if  $\langle x, y \rangle = \langle y, x \rangle$  for all  $x, y \in V$ . A symmetric bilinear form is nondegenerate if  $\langle a, x \rangle = 0$  for all  $x \in V$  implies  $a = 0$ . It is positive definite if  $\langle x, x \rangle > 0$  for any nonzero  $x \in V$ . A symmetric positive definite bilinear form is an inner product in  $V$ .

**Theorem 3.** Define a bilinear form on  $V = \mathbb{R}^n$  by  $\langle e_i, e_j \rangle = \delta_{ij}$ , where  $\{e_i\}_{i=1}^n$  is a basis in  $V$ , and  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  if  $i \neq j$ . Then  $\langle, \rangle$  is an inner product in  $V$ .

*Proof.* We must check that it is symmetric and positive definite:

- $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle = 1$  if  $i = j$  and  $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle = 0$  if  $i \neq j$
- $\langle e_i, e_i \rangle = 1$ .

□

**Definition 4.** A Euclidean space is a finite dimensional real linear space with an inner product.

**Theorem 5.** Any Euclidean  $n$ -dimensional space  $V$  has a basis  $\{e_i\}_{i=1}^n$  such that  $\langle e_i, e_j \rangle = \delta_{ij}$ .

*Proof.* First we will show that  $V$  has an orthogonal basis. Let  $\{u_1, \dots, u_n\}$  be a basis of  $V$ , and let  $v_1 = u_1$ ; note that  $v_1 \neq 0$ , and so  $\langle v_1, v_1 \rangle \neq 0$ . Also, let  $v_2$  be given by the following:

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1.$$

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The vectors  $v_1$  and  $v_2$  are orthogonal for

$$\langle v_2, v_1 \rangle = \langle u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1, v_1 \rangle = \langle u_2, v_1 \rangle - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} \langle v_1, v_1 \rangle = 0.$$

Note  $v_1$  and  $v_2$  are linearly independent, because  $u_1$  and  $u_2$  are basis elements and therefore linearly independent. We proceed by letting

$$v_j = u_j - \frac{\langle u_j, v_{j-1} \rangle}{\langle v_{j-1}, v_{j-1} \rangle} v_{j-1} - \dots - \frac{\langle u_j, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1.$$

Now, for  $j - 1 \geq k \geq 1$ ,

$$\langle v_j, v_k \rangle = \langle u_j, v_k \rangle - \frac{\langle u_j, v_{j-1} \rangle}{\langle v_{j-1}, v_{j-1} \rangle} \langle v_{j-1}, v_k \rangle - \dots - \frac{\langle u_j, v_k \rangle}{\langle v_k, v_k \rangle} \langle v_k, v_k \rangle - \dots - \frac{\langle u_j, v_1 \rangle}{\langle v_1, v_1 \rangle} \langle v_1, v_k \rangle = 0.$$

This equality holds by induction because  $\langle v_i, v_k \rangle = 0$  for  $i = j - 1, \dots, 1$  and  $i \neq k$ , and the remaining negative term on the right-hand side cancels with the first term for  $i = k$ . We cannot write the equation for  $v_j$  to give a linear relation between  $\{u_1, \dots, u_j\}$  because this is a subset of a basis. Thus,  $\{v_1, \dots, v_j\}$  are linearly independent. Continuing with induction until  $j = n$  produces an orthogonal basis for  $V$ . Note that all denominators in the above calculations are positive real numbers because  $v_i \neq 0$  for  $i = 1, \dots, n$ . Lastly, when we define

$$e_i = \frac{v_i}{\|v_i\|}, \quad i = 1, \dots, n,$$

then  $\|e_i\| = 1$  for all  $i$ , and  $\{e_1, \dots, e_n\}$  is an orthonormal basis for  $V$  such that  $\langle e_i, e_j \rangle = \delta_{ij}$ .  $\square$

Below  $V = \mathbb{R}^n$  is a Euclidean space with the inner product  $\langle, \rangle$ .

**Definition 6.** Two vectors  $x, y \in V$  are orthogonal if  $\langle x, y \rangle = 0$ . Two subspaces  $U, W \in V$  are orthogonal if  $\langle x, y \rangle = 0$  for all  $x \in U, y \in W$ .

**Theorem 7.** If  $U, W$  are orthogonal subspaces in  $V$ , then  $\dim(U) + \dim(W) = \dim(U + W)$ .

*Proof.* Let  $\{e_i\}_{i=1}^n$  be an orthonormal basis of  $V$ . A subset of these basis vectors will form a basis of  $U$ . Likewise, another subset of these basis vectors will form a basis of  $W$ . No basis vector of  $V$  can be a basis vector of both  $U$  and  $W$  because the two subspaces are orthogonal. We can see this by taking  $x$  to be a basis vector of  $U$  and taking  $y$  to be a basis vector of  $W$ . We now have

$$\langle x, y \rangle = \delta_{ij}$$

We know this is zero, so  $x$  and  $y$  cannot be the same. Since this applies to all  $x \in U$  and all  $y \in W$ , the two subspaces share none of the same

basis vectors. We can then form a basis of  $U + W$  by taking the basis of  $U$  plus the basis of  $W$ . Thus,  $\dim(U) + \dim(W) = \dim(U + W)$ .  $\square$

**Definition 8.** *The orthogonal complement of the subspace  $U \subset V$  is the subspace  $U^\perp = \{y \in V : \langle x, y \rangle = 0, \text{ for all } x \in U\}$ .*

**Definition 9.** *A hyperplane  $H_x \subset V$  is the orthogonal complement to the one-dimensional subspace in  $V$  spanned by  $x \in V$ .*

**Theorem 10.** (Cauchy-Schwartz). *For any  $x, y \in V$ ,*

$$\langle x, y \rangle^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle,$$

*and equality holds if and only if vectors  $x, y$  are linearly dependent.*

*Proof.* For all  $a$  and  $b$  in  $\mathbb{R}$  we have, by positive definiteness:

$$\langle ax - by, ax - by \rangle \geq 0$$

By bilinearity and symmetry this gives:

$$a^2 \langle x, x \rangle - 2ab \langle x, y \rangle + b^2 \langle y, y \rangle \geq 0$$

We can now set  $a = \langle x, y \rangle$  and  $b = \langle x, x \rangle$  and cancel the first term:

$$\langle x, x \rangle (-\langle x, y \rangle^2 + \langle x, x \rangle \langle y, y \rangle) \geq 0$$

If  $\langle x, x \rangle = 0$ , then by positive definiteness,  $x = 0$  and therefore  $\langle x, y \rangle = 0$  as well. Thus the equality holds in this case. If  $\langle x, x \rangle \neq 0$ , then it is positive and the theorem follows from the equality above.  $\square$

We will be interested in the linear mappings that respect inner products.

**Definition 11.** *An orthogonal operator in  $V$  is a linear automorphism  $f : V \rightarrow V$  such that  $\langle f(x), f(y) \rangle = \langle x, y \rangle$  for all  $x, y \in V$ .*

**Theorem 12.** *If  $f_1, f_2$  are orthogonal operators in  $V$ , then so are the inverses  $f_1^{-1}, f_2^{-1}$  and the composition  $f_1 \circ f_2$ . The identity mapping is orthogonal.*

*Proof.* Because an orthogonal operator  $f$  is one-to-one and onto, we know that for all  $a, b \in V$  there exists an  $x, y \in V$  such that  $x = f^{-1}(a)$  and  $y = f^{-1}(b)$ . Thus,  $f(x) = a$  and  $f(y) = b$ . By orthogonality,  $\langle f(x), f(y) \rangle = \langle x, y \rangle$ . This means that  $\langle a, b \rangle = \langle x, y \rangle$  and thus we conclude that  $\langle f^{-1}(a), f^{-1}(b) \rangle = \langle a, b \rangle$ .

The composition of  $f_1$  and  $f_2$  is  $\langle f_1 \circ f_2(x), f_1 \circ f_2(y) \rangle$ . Because  $f_1$  is orthogonal,  $\langle f_1 \circ f_2(x), f_1 \circ f_2(y) \rangle = \langle f_2(x), f_2(y) \rangle$ . Because  $f_2$  is also orthogonal,  $\langle f_2(x), f_2(y) \rangle = \langle x, y \rangle$ . The identity map is orthogonal by definition and this completes the proof.  $\square$

**Remark 13.** *The above theorem says that orthogonal operators in a Euclidean space form a group, i.e. a set closed with respect to compositions, contains an inverse to each element, and contains an identity operator.*

**Example 14.** *All orthogonal operators in  $\mathbb{R}^2$  can be expressed as  $2 \times 2$  matrices and they form a group with respect to the matrix multiplication.*

*The orthogonal operators of  $\mathbb{R}^2$  are all the reflections and rotations. The standard rotation matrix is the following*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

*and the standard reflection matrix is the following*

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

*The matrices of the above two types form the set of all orthogonal operators in  $\mathbb{R}^2$ . We know this because  $\langle Ax, Ay \rangle = \langle x, y \rangle$ . Thus,  $A^T A = I$  and so  $\det(A^2) = 1$  which implies  $\det(A) = \pm 1$ . The matrices with determinants plus or minus one are of the two forms shown above.*

Now we are ready to introduce the notion of a reflection in a Euclidean space. A reflection in  $V$  is a linear mapping  $s : V \rightarrow V$  which sends some nonzero vector  $\alpha \in V$  to its negative and fixes pointwise the hyperplane  $H_\alpha$  orthogonal to  $\alpha$ . To indicate this vector, we will write  $s = s_\alpha$ . The use of Greek letters for vectors is traditional in this context.

**Definition 15.** *A reflection in  $V$  with respect to a vector  $\alpha \in V$  is defined by the formula:*

$$s_\alpha(x) = x - \frac{2\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

**Theorem 16.** *With the above definition, we have:*

- (1)  $s_\alpha(\alpha) = -\alpha$  and  $s_\alpha(x) = x$  for any  $x \in H_\alpha$ ;
- (2)  $s_\alpha$  is an orthogonal operator;
- (3)  $s_\alpha^2 = Id$ .

*Proof.* (1) Using the formula, we know that,

$$s_\alpha(\alpha) = \alpha - \frac{2\langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle} (\alpha).$$

Since  $\frac{\langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 1$ , the formula simplifies to,

$$s_\alpha(\alpha) = \alpha - 2\alpha,$$

and thus  $s_\alpha(\alpha) = -\alpha$ .

For  $x \in H_\alpha$ , we know that  $\langle x, \alpha \rangle = 0$ . We can now write,

$$s_\alpha(x) = x - \frac{2(0)}{\langle \alpha, \alpha \rangle} \alpha,$$

and so  $s_\alpha(x) = x$ .

- (2) By Theorem 5, we can write any  $x \in V$  as a sum of basis vectors  $\{e_i\}_{i=1}^n$  such that  $\langle e_i, e_j \rangle = \delta_{ij}$ . We can also choose this basis so that  $e_1$  is in the direction of  $\alpha$  and the others are orthogonal to  $\alpha$ . Because of linearity, it suffices to check that  $\langle s_\alpha(e_i), s_\alpha(e_j) \rangle = \langle e_i, e_j \rangle$ .

$$s_\alpha(e_i) = e_i - \frac{2\langle e_i, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

If  $e_i \neq \alpha$  then it must be orthogonal to  $\alpha$ .

$$s_\alpha(e_i) = e_i - \frac{2(0)}{\langle \alpha, \alpha \rangle} \alpha = e_i$$

If  $e_i = e_1 = \alpha$  then by part one  $s_\alpha(e_1) = -e_1$ . We can now conclude that  $\langle s_\alpha(e_i), s_\alpha(e_j) \rangle = \langle \pm e_i, \pm e_j \rangle$ . It is easy to check that  $\langle \pm e_i, \pm e_j \rangle = \langle e_i, e_j \rangle = \delta_{ij}$ .

- (3) By the same reasoning as part two, it suffices to check that  $s_\alpha^2(e_i) = e_i$  for any basis vector  $e_i$ . Also from part two, we know that  $s_\alpha(e_i)$  is either equal to  $e_i$  or  $-e_i$ . If  $s_\alpha(e_i) = e_i$ , then  $e_i = \alpha$  and clearly  $s_\alpha^2(e_i) = e_i$ . If  $s_\alpha(e_i) = -e_i$ , then  $s_\alpha^2(e_i) = -(-e_i) = e_i$  and this completes the proof.  $\square$

Therefore, reflections generate a group: their compositions are orthogonal operators by Theorem 12, and an inverse of a reflection is equal to itself by Theorem 16. Below we consider some basic examples of subgroups of orthogonal operators obtained by repeated application of reflections.

**Example 17.** Consider the group  $S_n$  of permutations of  $n$  numbers. It is generated by transpositions,  $t_{ij}$  where  $i \neq j$  are two numbers between 1 and  $n$ , and  $t_{ij}$  sends  $i$  to  $j$  and  $j$  to  $i$ , while preserving all other numbers. The compositions of all such transpositions form  $S_n$ . Define a set of linear mappings  $T_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  in an orthonormal basis  $\{e_i\}_{i=1}^n$  by

$$T_{ij}e_i = e_j; \quad T_{ij}e_j = e_i; \quad T_{ij}e_k = e_k, k \neq i, j.$$

Then, since any element  $\sigma \in S_n$  is a composition of transpositions, it defines a linear automorphism of  $\mathbb{R}^n$  equal to the composition of the linear mappings defined above.

- (1)  $T_{ij}$  acts as a reflection with respect to the vector  $e_i - e_j \in \mathbb{R}^n$ .

*Proof.* We begin with our definition of a reflection.

$$s_\alpha(x) = x - \frac{2\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

$$s_{e_i - e_j}(x) = x - \frac{2\langle x, e_i - e_j \rangle}{\langle e_i - e_j, e_i - e_j \rangle} (e_i - e_j).$$

Because of linearity, it suffices to check the following:

(a)

$$s_{e_i - e_j}(e_k) = e_k, k \neq i, j$$

(b)

$$s_{e_i - e_j}(e_k) = e_j, k = i$$

(c)

$$s_{e_i - e_j}(e_k) = e_i, k = j$$

The third item follows from the second by symmetry, so we must only check the first two conditions outlined above. Note that  $\langle e_i - e_j, e_i - e_j \rangle = 2$  because

$$\langle e_i - e_j, e_i - e_j \rangle = \langle e_i, e_i - e_j \rangle - \langle e_j, e_i - e_j \rangle = \langle e_i, e_i \rangle - \langle e_i, e_j \rangle - \langle e_j, e_i \rangle + \langle e_j, e_j \rangle = 1 + 0 + 0 + 1 = 2$$

and so our formula simplifies to

$$s_{e_i - e_j}(e_k) = e_k - \langle e_k, e_i - e_j \rangle (e_i - e_j).$$

Suppose  $k$  does not equal  $i$  or  $j$  for condition (a).

$$\langle e_k, e_i - e_j \rangle = \langle e_k, e_i \rangle - \langle e_k, e_j \rangle = 0 + 0 = 0$$

$$s_{e_i - e_j}(e_k) = e_k.$$

Suppose  $k$  equals  $i$  for condition (b).

$$\langle e_i, e_i - e_j \rangle = \langle e_i, e_i \rangle - \langle e_i, e_j \rangle = 1 + 0 = 1$$

$$s_{e_i - e_j}(e_i) = e_i - (e_i - e_j) = e_j.$$

Thus,  $T_{ij}$  does indeed act like a reflection with respect to the vector  $e_i - e_j \in \mathbb{R}^n$ .  $\square$

- (2) Any element  $\sigma$  of  $S_n$  fixes pointwise the line in  $\mathbb{R}^n$  spanned by  $e_1 + e_2 + \dots + e_n$ .

*Proof.* The line described can be written as a variable times the sum of the basis vectors.

$$c(e_1 + e_2 + \dots + e_n), c \in \mathbb{R}.$$

In other words, the line is the set of points equidistant from each basis vector. Because vector addition is commutative, it matters not in which order the basis vectors are added. Since we know that elements of  $S_n$  simply switch basis vectors around, the line described remains unchanged.  $\square$

- (3) Let  $n = 3$ . There are six elements  $S_3$  in  $\mathbb{R}^3$ . If we call the basis vectors  $x, y, z$  then the six elements are:  $T_{xy}, T_{xz}, T_{yz}, T_{xy}T_{xz}, T_{xy}T_{yz}, T_{xx}$ . There is a special plane  $U$  orthogonal to  $x + y + z$  such that  $T_{ij}[U] = U$  for all  $i$  and  $j$ . It can be said that  $U$  is closed under transpositions, because any point in  $U$  remains in  $U$  after a transposition (although the point may be in a different place, it is still in the plane  $U$ ). Because  $U$  is closed, it can be thought of as  $\mathbb{R}^2$  and the elements of  $S_3$  correspond to the reflection and rotation matrices mentioned earlier in Example 14.  $T_{xy}$  reflects the plane  $U$  through the plane  $x = y$ .  $T_{xz}$  and  $T_{yz}$  reflect the plane  $U$  through the planes  $x = z$  and  $y = z$  respectively. These three transpositions correspond to reflections in  $\mathbb{R}^2$  about the line where the two planes intersect. The other three elements of  $S_3$  rotate  $U$  about the line described above in part (2).  $T_{xy}T_{xz}$  rotates  $U$  120 degrees clockwise.  $T_{xy}T_{yz}$  rotates  $U$  120 degrees counter-clockwise. Finally,  $T_{xx}$  is a rotation by 360 degrees, also known as the identity map.

**Remark 18.** Notice that the product of two reflections is a rotation. As seen above in the plane  $U$ , a transposition is a reflection, and the composition of any two transpositions is a rotation.

**Example 19.** The action of  $S_n$  in  $\mathbb{R}^n$  described above can be composed with the reflections  $\{P_i\}_{i=1}^n$ , sending  $e_i$  to its negative and fixing all other elements of the basis  $e_k, k \neq i$ .

- (1) The obtained set of orthogonal operators has no identity element.

*Proof.* If  $T_{ij}P_k$  were an identity element, then  $P_k$  would have to be the inverse of  $T_{ij}$ . However,  $P_k$  is its own inverse, so  $P_k$  must equal  $T_{ij}$ . There is no transposition  $T_{ij}$  that is equal to a reflection  $P_k$  because  $T_{ij}$  acts as a reflection with respect to the vector  $e_i - e_j$  and  $P_k$  acts as a reflection with respect to the

vector  $e_k$ . Since there are no zero basis vectors,  $e_i - e_j$  never equals  $e_k$ .

There is no overlap between the  $T_{ij}$ 's and the  $P_k$ 's and thus there cannot be an identity element.  $\square$

- (2) *Four distinct orthogonal operators can be constructed in this way for  $n = 2$  and 18 can be constructed for  $n = 3$ .*

*In  $\mathbb{R}^2$  there exist two reflections;  $P_x, P_y$ , and two transpositions;  $T_{xy}$ , and  $T_{xx}$  (the identity). These can be composed in  $2 * 2 = 4$  ways.*

*In  $\mathbb{R}^3$  there are three reflections and six elements of  $S_n$ , and these can be composed in  $3 * 6 = 18$  ways.*

- (3) *Consider the case where  $n = 2$ . The operators correspond to the matrices of orthogonal operators in  $\mathbb{R}^2$  as listed in Example 14. The operator  $T_{xx}P_x$  is a reflection about the  $y$ -axis. The operator  $T_{xx}P_y$  is a reflection about the  $y$ -axis. The operator  $T_{xy}P_x$  is a rotation counter-clockwise by 270 degrees. The operator  $T_{xy}P_y$  is a rotation counter-clockwise by 90 degrees.*

**Remark 20.** *The two examples above correspond to the series  $A_{n-1}$  and  $B_n$  in the classification of finite reflection groups.*