18.099/18.06CI - HOMEWORK 4

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Problem 1.

(a) Given a finite dimensional linear space L and a subspace $L_1 \subset L$ we want to prove that there exists a subspace $L_2 \subset L$ such that $L_1 \oplus L_2 = L$. Furthermore, we want to prove that the dimensions of all such direct complements to L_1 coincide.

Let $\dim L = l$. We pick a basis $\{e_i\}_{i=1}^{l_1}$ for L_1 and note that $\dim L_1 = l_1$. We then use the basis extension theorem and extend the basis of L_1 to the basis of L. Thus, $\{e_1, \ldots, e_{l_1}, e_{l_1+1} \ldots, e_l\}$ spans L. Let $L_2 =$ span $\{e_i\}_{i=l_1+1}^{l}$. Since, $L_1 \cap L_2 = 0$ and $L_1 + L_2 = L$ we note that L_2 is a direct complement to L_1. Furthermore, $\dim L_2 = l - l_1$.

Let L_2' be a direct complement to L_1 and $\{e_i'\}_{i=1}^{l_2'}$ be a basis for L_2'. We extend the basis of L_2' to L by using the basis vectors of L_1. Thus, $\{e_1', \ldots, e_{l_2'}', e_1, \ldots, e_{l_1}\}$ spans L. Hence, it must be that $\dim L_2' = l - l_1$. However, this is the same as $\dim L_2$. Thus, dimensions of all direct complements of L_1 coincide.

(b) Given $F : L \mapsto M$, we first show that $\operatorname{ind} F = \dim(\operatorname{coker} F) - \dim(\ker F)$ is well defined. We note that since part (a) defined direct complement only for finite dimensional spaces we restrict our attention to a finite dimensional M.

Using the result from part (a) we find that $\operatorname{coker} F$, a direct complement to $\operatorname{Im} F \subset M$, always exists and has a finite dimension. Furthermore, $\ker F$ also always exists and has a well defined dimension. We can then take $\dim \operatorname{coker} F = c$ and $\dim \ker F = k$. Hence, $\operatorname{ind} F = c - k$. We note that c is always a non-negative integer and k can be either a non-negative integer or infinity depending on the dimension of L. Thus, $\operatorname{ind} F$ is well defined.

For finite dimensional M and L. Let $\dim M = m$ and $\dim \operatorname{Im} F = i$. Then $\dim \operatorname{coker} F = m - i$. Further, let $\dim L = l$. Then $\dim \ker F = \dim L - \dim \operatorname{Im} F = l - i$. Thus, $\operatorname{ind} F = (m-i)-(l-i) = m - l = \dim M - \dim L$

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(c) If $\dim M = \dim L = n$. Then $\operatorname{ind} F = 0$ and also $\dim \operatorname{coker} F = \dim \ker F$. If $\ker F = 0$ then also $\operatorname{coker} F = 0$ and the system of linear equations always has a solution, while the system with a zero r.h.s has no nontrivial solution.

Problem 2.

We want to show that all triples of non-coplanar, pairwise distinct lines through zero in $\mathbb{R}^3$ are identically arranged. Let $\{e_1, e_2, e_3\}$ be a basis for $\mathbb{R}^3$ and let $v_i = a_{1i}e_1 + a_{2i}e_2 + a_{3i}e_3$ for $i = 1, 2, 3$ be direction vectors for three non-coplanar pairwise distinct lines in $\mathbb{R}^3$. Further, let $v'_i = a'_{1i}e_1 + a'_{2i}e_2 + a'_{3i}e_3$ for $i = 1, 2, 3$ be the direction vectors for a second set of non-coplanar pairwise distinct lines in $\mathbb{R}^3$. For v_i and v'_i to be identically arranged we must find a linear map f such that $f(v_i) = v'_i$ for $i = 1, 2, 3$. This is equivalent to finding a matrix T such that $T(a_{ij}) = (a'_{ij})$. Since the three lines are linearly independent we can invert the matrix of coefficients. Thus, $T = (a'_{ij})(a_{ij})^{-1}$ and three such lines are identically arranged.

To consider the arrangements of four such lines we note that direction vectors for three such lines span $\mathbb{R}^3$ and thus we express the direction vector for the fourth line as a linear combination of the first three. Namely, $v_4 = b_1v_1 + b_2v_2 + b_3v_3$ and $v'_4 = b'_1v'_1 + b'_2v'_2 + b'_3v'_3$. Further, $T(v_4) = b_1T(v_1) + b_2T(v_2)+b_3T(v_3)$. Hence, if we add a scaling factor to the first three direction vectors $T(v_i) = \frac{b'_i}{b_i}v'_i$ for $i = 1, 2, 3$ it follows that $T(v_4) = v'_4$ and thus all quadruples of such lines are identically arranged.