18.099/18.06CI - HOMEWORK 3

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Problem 1.

Let L and M be finite dimensional linear spaces and let $f : L \mapsto M$ be a linear map. We want to show that Imf and $\ker f$ are finite dimensional and that dim Im $f + \dim \ker f = \dim L$. We assume that $\dim L = n$. Let $\{b_j\}$ be a basis for kernel of f. By the Basis Extension Theorem we know that there are linearly independent vectors $\{c_k\}$ such that the ordered set $\{a_i\} = \{b_j, c_k\}$ forms a basis for L. Since the total number of basis elements for L is finite, it must be that $\ker f \subseteq L$ is also finite dimensional. Thus, $\dim \ker f = l \leq n$.

Next we show that $\{f(a_i)\}_{i=l+1}^{n}$ span Imf. We choose $v \in$ Imf. Since $\{a_i\}$ spans L, there exist scalars d_i such that $v = f\left(\sum_{i=1}^{n} d_i a_i\right)$. Noting that $f(a_i) = 0,\ 1 \leq i \leq l$, we have $v = \sum_{i=l+1}^{n} c_i f(a_i)$. Thus, $\{f(a_i)\}_{i=l+1}^{n}$ spans Imf.

To verify the linear independence of $\{f(a_i)\}_{i=l+1}^{n}$ we consider the linear combination $\sum_{i=l+1}^{n} c_i f(a_i) = 0$. Then by linearity $f\left(\sum_{i=l+1}^{n} c_i a_i\right) = 0$. Since a_i are linearly independent it must be that all $c_i = 0$. Hence all $\{f(a_i)\}_{i=l+1}^{n}$ are linearly independent and we can consider them a basis for Imf. This also confirms that Imf is finite dimensional.

Hence, $\dim \text{Im} f + \dim \ker f = \dim L$.

Problem 2.

(a) $\{f \in \mathcal{L}(V, V) \mid \ \dim \text{Im} f = 0\}$

This is a subspace. The defining condition of this subset is equivalent to $f = 0$. Thus for all $v \in V$, $f(v) = 0$. By linearity, this is also true for any combination of such linear maps.

Considering the matrix representation of this subspace of linear maps, we conclude that it is isomorphic to the subspace of zero matrices.

(b) $\{f \in \mathcal{L}(V, V) \mid \ \dim \ker f = 0\}$

This is not a subspace, unless $\dim V = 0$. We prove this by contradiction. Suppose $v \in V$. Then necessarily for f in the subset, $f(v) \neq 0$. Further, by linearity it must be that $kf(v) \neq 0, k \in \mathbb{F}$. We pick $k = 0$ and then for any vector $v \in V$, the action of kf on v gives

Date: March 8, 2004.

0, independent of v. Thus $\dim\ker kf \neq 0$. However, if $\dim V = 0$ then also $\dim\ker f = 0$. In this case $V \subseteq \{0\}$ and necessarily $f = 0$.

Formulated in terms of matrices, we can consider this subset of linear transforms isomorphic with the set of matrices with rank equal to the dimension of V. This is not a subspace since multiplication of any such matrix by 0 will result in a matrix that has rank less than the dimension of V unless the dimension of V is 0.

(c) $\{f \in \mathcal{L}(V,V) \mid \ \dim \mathrm{Im} f < \dim V\}$.

This is not a subspace, unless $\dim V = 1$. We prove this by contradiction. Let $\dim V \geq 2$ and let $\{a_i\}_{i=1}^n$ be a basis for V. Further, let $f_i(a_j) = a_j$, for $i = j$ and $f_i(a_j) = 0$, for $i \neq j$. Then for $v = \sum_{i=1}^n c_i a_i$, $f_i(v) = c_i a_i$. Thus, $\dim \mathrm{Im} f_i = 1 < \dim V$. However, $\left(\sum_{i=1}^n f_i\right) v = v$ and $\dim \mathrm{Im}\left(\sum_{i=1}^n f_i\right) = n = \dim V$. For $\dim V = 1$ it must be that $f = 0, \dim \mathrm{Im} f = 0$ and proof follows as in part (a). We cannot have $\dim V = 0$, since then $\dim \mathrm{Im} f = \dim V$.

We can consider this subset of linear transforms isomorphic to the set of matrices with rank less than the dimension of V. For $\dim V > 1$ we consider a subset of such linearly independent matrices of rank 1 and note that the sum of these matrices may result in a matrix of rank $\dim V$. In case $\dim V = 1$ our subset is a subspace of zero matrices as described in part (a).

Problem 3.

(a) We want to prove that there are no finite square matrices such that $XY - YX = I$. By contradiction, suppose we could find such X and Y. Let $X, Y, I \in \mathbb{F}^{m\times m}$. Then trace $(XY - YX)$ = trace $I = m$. However,

$$\begin{aligned}
\text{trace } (XY - YX) &= \sum_{i=1}^m (XY - YX)_{ii} \\
&= \sum_{i=1}^m \left(\sum_{k=1}^m (X)_{ik}(Y)_{ki} - \sum_{k=1}^m (Y)_{ik}(X)_{ki} \right) \\
&= \sum_{i=1}^m \sum_{k=1}^m (X)_{ik}(Y)_{ki} - \sum_{i=1}^m \sum_{k=1}^m (Y)_{ik}(X)_{ki} \\
&= 0 \neq m
\end{aligned}$$

Thus, it is impossible to find finite square matrices that satisfy $XY - YX = I$.

(b) Let $P = \sum_{k=0}^n c_k x^k$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x} P = \sum_{k=0}^n k c_k x^{k-1} \quad \text{and} \quad xP = \sum_{k=0}^n c_k x^{k+1}.$$

Both of these maps are linear:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(aP+P') &= \frac{\mathrm{d}}{\mathrm{d}x}(a\sum_{k=0}^{n} c_k x^k + \sum_{k=0}^{n} c'_k x^k) \\ &= a\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{n} c_k x^k + \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{n} c'_k x^k \\ &= a\frac{\mathrm{d}}{\mathrm{d}x}P + \frac{\mathrm{d}}{\mathrm{d}x}P'\end{aligned}$$

$$\begin{aligned}x(aP+P') &= x(a\sum_{k=0}^{n} c_k x^k + \sum_{k=0}^{n} c'_k x^k) \\ &= ax\sum_{k=0}^{n} c_k x^k + x\sum_{k=0}^{n} c'_k x^k \\ &= axP + xP'\end{aligned}$$

We calculate the compositions of these two maps:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\circ x\right)P = \frac{\mathrm{d}}{\mathrm{d}x}(xP) = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{n} c_k x^{k+1} = \sum_{k=0}^{n}(k+1)c_k x^k$$

$$\left(x\circ\frac{\mathrm{d}}{\mathrm{d}x}\right)P = x\left(\frac{\mathrm{d}}{\mathrm{d}x}P\right) = x\sum_{k=0}^{n} k c_k x^{k-1} = \sum_{k=0}^{n} k c_k x^k.$$

Thus by linearity

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}x}\circ x - x\circ\frac{\mathrm{d}}{\mathrm{d}x}\right)P &= \sum_{k=0}^{n}(k+1)c_k x^k - \sum_{k=0}^{n} k c_k x^k \\ &= \sum_{k=0}^{n} c_k x^k = I\,P = P\end{aligned}$$

(c) We introduce the concept of matrices with infinitely many rows and columns. To define multiplication of two such matrices, we restrict our attention to matrices with finitely many non-zero elements on any given row or column. Then for any given row $\{a_{ij}\}_{j=1}^{\infty}$ there exists a number N_i such that $a_{ij} = 0, \text{for } j > N_i$. Similarily for any given column $\{a_{ij}\}_{i=1}^{\infty}$ there exists a number M_j such that $a_{ij} = 0, \text{for } i > M_j$. Multiplication of two infinite matrices A and B is then defined as

$$(AB)_{ij} = \sum_{k=1}^{\min(A_{N_i}, B_{M_j})} (A)_{ik}(B)_{kj}$$

In order to construct the matrix representations of linear transforms x and $\frac{\mathrm{d}}{\mathrm{d}x}$ we consider their actions on the standard basis element x^k. For $k \geq 0$, $x(x^k) = x^{k+1}$. Similarily for $k > 0$, $\frac{\mathrm{d}}{\mathrm{d}x}(x^k) = kx^{k-1}$, $k = 0$, $\frac{\mathrm{d}}{\mathrm{d}x}(x^0) = 0$. Hence,

$$X = A_{\frac{\mathrm{d}}{\mathrm{d}x}} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix} \quad Y = A_x = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

By matrix multiplication:

$$XY = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 2 & 0 & \dots \\ 0 & 0 & 3 & \\ \vdots & \vdots & & \ddots \end{pmatrix} \quad YX = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 2 & \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

Thus:

$$XY - YX = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \end{pmatrix} = I.$$

Hence, X and Y as defined above are a solution to $XY - YX = I$.