18.099/18.06CI - HOMEWORK 1

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Problem 1.

The field $\mathbb{Q}$ is a linear space over $\mathbb{Q}$, but not over $\mathbb{R}$. In order to prove the latter statement by contradiction, suppose that $\mathbb{Q}$ is a linear space over $\mathbb{R}$. For $v \neq 0, v \in \mathbb{Q}$, $f, g \in \mathbb{R}, fv = gv \Rightarrow f = g$ and $fv \neq gv \Rightarrow f \neq g$. This then would give a one-to-one map from $\mathbb{R}$ to $\mathbb{Q}$ defined as $f \mapsto fv$. However, since the cardinality of $\mathbb{R}$ is greater than the cardinality of $\mathbb{Q}$, there cannot be a one-to-one map from $\mathbb{R}$ to $\mathbb{Q}$. Thus, by contradiction, $\mathbb{Q}$ is not a linear space over $\mathbb{R}$. Conversely, one-to-one map from $\mathbb{Q}$ to $\mathbb{R}$ can be defined as $q \mapsto qr, q \in \mathbb{Q}, r \in \mathbb{R}$. Hence, by restricting the coefficients from $\mathbb{R}$ to $\mathbb{Q}$, any linear space over $\mathbb{R}$ becomes a linear space over $\mathbb{Q}$.

Problem 2.

(a) Yes, sequences with only finitely many nonzero elements are a subspace of A. Let S be all the infinite sequences over $\mathbb{R}$ with finitely many non-zero terms and let $a, b \in S,\ k \in \mathbb{R}$. It is clear that $a + kb \in S$ since the number of non-zero terms will still be finite.

(b) No, sequences with only finitely many zero terms are not a subspace of A. Let S be all the infinite sequences over $\mathbb{R}$ with only finitely many zero terms and let $a \in S$. Since $0 \cdot a = 0 \notin S$, S is not a linear space.

(c) Yes, Cauchy sequences are a subspace of A. Let S be the set of all Cauchy sequences and $a, b \in S$. Suppose ε_{ab} is given and choose $\varepsilon_a, \varepsilon_b > 0$ such that $\varepsilon_{ab} = \varepsilon_a + \varepsilon_b$. Find $N_a, N_b \in \mathbb{R}$ such that $|a_n - a_m| < \varepsilon_a$ for all $m, n > N_a$ (similarily for b). We need to locate N_{ab} such that $|(a_n + b_n) - (a_m + b_m)| < \varepsilon_{ab}$ for all $m, n > N_{ab}$. From triangle inequality $|A + B| \leq |A| + |B|$. Hence, for $N_{ab} = max(N_a, N_b), |(a_n - a_m) + (b_n - b_m)| \leq \varepsilon_a + \varepsilon_b = \varepsilon_{ab}$.

(d) Yes, the sequences, for which the sum of the squares of the elements converges, is a subspace of A. Let S be the set of all the infinite sequences $\{a_i\}_{i=1}^{\infty}, a_i \in \mathbb{R}$ for which $\sum_{i=1}^{\infty} a_i^2$ converges. Then for $a, b \in S,\ a + b \in S :\ \sum(a_i + b_i)^2 = \sum a_i^2 + \sum b_i^2 + 2\sum a_i b_i$. By Cauchy-Schwarz $(\sum x_i^2) \cdot (\sum y_i^2) \geq (\sum a_i b_i)^2$. Also, for $k \in \mathbb{R}, ka \in S : \sum(ka_i)^2 = k^2 \cdot \sum a_i^2$. Therefore, S is a linear space.

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