18.099 - 18.06CI.

HW-4

Due on Monday, March 15 in class. First draft due by Thursday, March 11.

(1) (a) A *direct complement* to a subspace L_1 in a finite dimensional space L is a subspace $L_2 \subset L$ such that $L = L_1 \oplus L_2$. Prove that for any subspace L_1 a direct complement exists, and the dimensions of any two direct complements of L_1 in L coincide.

(b) For a linear map $F : L \to M$, let $\text{coker}F$ be a direct complement of $\text{Im}F$ in M. Define the *index* of the map F by

$$\text{ind}F = \dim(\text{coker}F) - \dim(\text{ker}F).$$

Check using (a) that $\text{ind}F$ is well-defined. Prove that if L and M are finite dimensional, $\text{ind}F$ depends only on the dimensions of L and M:

$$\text{ind}F = \dim(M) - \dim(L).$$

(c) Set $\dim(M) = \dim(L) = n$. What can you deduce from (b) about systems of n linear equations in n variables?

(2) Two ordered n-tuples of subspaces $\langle L_1, L_2, \dots, L_n \rangle$ and $\langle L'_1, L'_2, \dots L'_n \rangle$ in a finite dimensional L are *identically arranged* if there exists a linear automorphism (bijecitve linear map from a space to itself) $F : L \to L$ such that $F(L_i) = L'_i$ for all $i = 1, \dots n$. Show that all triples of non-coplanar, pairwise distinct lines through zero in $\mathbb{R}^3$ are identically arranged. Classify the arrangements of quadruples of such lines in $\mathbb{R}^3$.