18.099 - 18.06CI.

HW-3

Due on Monday, March 8 in class. First draft due on Thursday, Feb 26. First $\LaTeX$draft due on Thursday, March 4.

(1) Let L and M be finite dimensional linear spaces and let $f : L \to M$ be a linear map. Prove that the subspaces $\ker f$ and $\mathrm{Im} f$ are finite dimensional and

$$\dim \ker f + \dim \mathrm{Im} f = \dim L.$$

Hint: Use the extension of a basis theorem.

(2) Consider the space $\mathcal{L}(V, V)$ of linear maps from a finite dimensional real linear space V to itself. Which of the following subsets of $\mathcal{L}(V, V)$ are linear spaces:
   (a) $\{f \in \mathcal{L}(V, V) \mid \ \dim \mathrm{Im} f = 0\}$;
   (b) $\{f \in \mathcal{L}(V, V) \mid \ \dim \ker f = 0\}$;
   (c) $\{f \in \mathcal{L}(V, V) \mid \ \dim \mathrm{Im} f < \dim V\}$.

   Reformulate the obtained results in terms of matrices.

(3) (a) Show that there are no finite square real or complex matrices X and Y such that $XY - YX = I$, where I is the identity matrix.
   (b) Check that multiplication by x and differentiation $\frac{d}{dx}$ are linear maps on the (real or complex) space $\mathcal{P}$ of all polynomials in one variable x. Compute the action of $\frac{d}{dx} \circ x - x \circ \frac{d}{dx}$ in $\mathcal{P}$. Here "$\circ$" denotes the composition of two maps.
   (c) Introduce the concept of a matrix with infinitely many rows and columns. Define the multiplication for infinite matrices in which each column and each row has a finite number of non-zero elements. Using (b), find a solution for $XY - YX = I$ in terms of infinite matrices.