

# Confidence Intervals: Three Views

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## 1 Learning Goals

1. Be able to produce  $z$ ,  $t$  and  $\chi^2$  confidence intervals based on the corresponding standardized statistics.
2. Be able to use a hypothesis test to construct a confidence interval for an unknown parameter.
3. Refuse to answer questions that ask, in essence, ‘given a confidence interval what is the probability or odds that it contains the true value of the unknown parameter?’

## 2 Introduction

Our approach to confidence intervals in the previous reading was a combination of standardized statistics and hypothesis testing. Today we will consider each of these perspectives separately, as well as introduce a third formal viewpoint. Each provides its own insight.

1. **Standardized statistic.** Most confidence intervals are based on standardized statistics with known distributions like  $z$ ,  $t$  or  $\chi^2$ . This provides a straightforward way to construct and interpret confidence intervals as a point estimate plus or minus some error.
2. **Hypothesis testing.** Confidence intervals may also be constructed from hypothesis tests. In cases where we don’t have a standardized statistic this method will still work. It agrees with the standardized statistic approach in cases where they both apply.

This view connects the notions of significance level  $\alpha$  for hypothesis testing and confidence level  $1 - \alpha$  for confidence intervals; we will see that in both cases  $\alpha$  is the probability of making a ‘type 1’ error. This gives some insight into the use of the word confidence. This view also helps to emphasize the frequentist nature of confidence intervals.

3. **Formal.** The formal definition of confidence intervals is perfectly precise and general. In a mathematical sense it gives insight into the inner workings of confidence intervals. However, because it is so general it sometimes leads to confidence intervals without useful properties. We will not dwell on this approach. We offer it mainly for those who are interested.

## 3 Confidence intervals via standardized statistics

The strategy here is essentially the same as in the previous reading. Assuming normal data we have what we called standardized statistics like the standardized mean, Studentized mean, and standardized variance. These statistics have well known distributions which depend on hypothesized values of  $\mu$  and  $\sigma$ . We then use algebra to produce confidence intervals for  $\mu$  or  $\sigma$ .

Don't let the algebraic details distract you from the essentially simple idea underlying confidence intervals: we start with a standardized statistic (e.g.,  $z$ ,  $t$  or  $\chi^2$ ) and use some algebra to get an interval that depends only on the data and *known* parameters.

### 3.1 $z$ -confidence intervals for $\mu$ : normal data with known $\sigma$

$z$ -confidence intervals for the mean of normal data are based on the standardized mean, i.e. the  $z$ -statistic. We start with  $n$  independent normal samples

$$x_1, x_2, \dots, x_n \sim N(\mu, \sigma^2).$$

We assume that  $\mu$  is the unknown parameter of interest and  $\sigma$  is known.

We know that the standardized mean is standard normal:

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

For the standard normal critical value  $z_{\alpha/2}$  we have:  $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$ .

Thus,

$$P\left(-z_{\alpha/2} < \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} < z_{\alpha/2} \mid \mu\right) = 1 - \alpha$$

A little bit of algebra puts this in the form of an interval around  $\mu$ :

$$P\left(\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \mid \mu\right) = 1 - \alpha$$

We can emphasize that the interval depends only on the statistic  $\bar{x}$  and the known value  $\sigma$  by writing this as

$$P\left(\left[\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right] \text{ contains } \mu \mid \mu\right) = 1 - \alpha.$$

This is the  $(1 - \alpha)$   $z$ -confidence interval for  $\mu$ . We often write it using the shorthand

$$\bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

Think of it as  $\bar{x} \pm$  error.

Make sure you notice that the probabilities are conditioned on  $\mu$ . As with all frequentist statistics, we have to fix hypothesized values of the parameters in order to compute probabilities.

### 3.2 $t$ -confidence intervals for $\mu$ : normal data with unknown $\mu$ and $\sigma$

$t$ -confidence intervals for the mean of normal data are based on the Studentized mean, i.e. the  $t$ -statistic.

Again we have  $x_1, x_2, \dots, x_n \sim N(\mu, \sigma^2)$ , but now we assume both  $\mu$  and  $\sigma$  are unknown. We know that the Studentized mean follows a Student  $t$  distribution with  $n - 1$  degrees of freedom. That is,

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} \sim t(n - 1),$$

where  $s^2$  is the sample variance.

Now all we have to do is replace the standardized mean by the Studentized mean and the same logic gives us the  $t$ -confidence interval: start with

$$P\left(-t_{\alpha/2} < \frac{\bar{x} - \mu}{s/\sqrt{n}} < t_{\alpha/2} \mid \mu\right) = 1 - \alpha.$$

A little bit of algebra isolates  $\mu$  in the middle of an interval:

$$P\left(\bar{x} - t_{\alpha/2} \cdot \frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2} \cdot \frac{s}{\sqrt{n}} \mid \mu\right) = 1 - \alpha$$

We can emphasize that the interval depends only on the statistics  $\bar{x}$  and  $s$  by writing this as

$$P\left(\left[\bar{x} - t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}\right] \text{ contains } \mu \mid \mu\right) = 1 - \alpha.$$

This is the  $(1 - \alpha)$   $t$ -confidence interval for  $\mu$ . We often write it using the shorthand

$$\bar{x} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

Think of it as  $\bar{x} \pm$  error.

### 3.3 $\chi^2$ -confidence intervals for $\sigma^2$ : normal data with unknown $\mu$ and $\sigma$

You guessed it:  $\chi^2$ -confidence intervals for the variance of normal data are based on the ‘standardized variance’, i.e. the  $\chi^2$ -statistic.

We follow the same logic as above to get a  $\chi^2$ -confidence interval for  $\sigma^2$ . Because this is the third time through it we’ll move a little more quickly.

We assume we have  $n$  independent normal samples:  $x_1, x_2, \dots, x_n \sim N(\mu, \sigma^2)$ . We assume that  $\mu$  and  $\sigma$  are both unknown.

We know that the  $X^2$  statistic follows a  $\chi^2$  distribution with  $n - 1$  degrees of freedom:

$$X^2 = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1).$$

For  $Z$  and  $t$  we used, without comment, the symmetry of the distributions to replace  $z_{1-\alpha/2}$  by  $-z_{\alpha/2}$  and  $t_{1-\alpha/2}$  by  $-t_{\alpha/2}$ . Because the  $\chi^2$  distribution is not symmetric we need to be explicit about the critical values on both the left and the right. That is,

$$P(c_{1-\alpha/2} < X^2 < c_{\alpha/2}) = 1 - \alpha,$$

where  $c_{\alpha/2}$  and  $c_{1-\alpha/2}$  are *right tail* critical values. Thus,

$$P\left(c_{1-\alpha/2} < \frac{(n-1)s^2}{\sigma^2} < c_{\alpha/2} \mid \sigma\right) = 1 - \alpha$$

A little bit of algebra puts this in the form of an interval around  $\sigma^2$ :

$$P\left(\frac{(n-1)s^2}{c_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{c_{1-\alpha/2}} \mid \sigma\right) = 1 - \alpha$$

We can emphasize that the interval depends only on the statistic  $s^2$  by writing this as

$$P\left(\left[\frac{(n-1)s^2}{c_{\alpha/2}}, \frac{(n-1)s^2}{c_{1-\alpha/2}}\right] \text{ contains } \sigma^2 \mid \sigma^2\right) = 1 - \alpha.$$

This is the  $(1 - \alpha)$   $\chi^2$ -confidence interval for  $\sigma^2$ .

## 4 Confidence intervals via hypothesis testing

Suppose we have data drawn from a distribution with a parameter  $\theta$  whose value is unknown. A significance test for the value  $\theta$  has the following short description.

1. Set the null hypothesis  $H_0 : \theta = \theta_0$  for some special value  $\theta_0$ , e.g. we often have  $H_0 : \theta = 0$ .
2. Use the data to compute the value of a test statistic, call it  $x$ .
3. If  $x$  is far enough into the tail of the null distribution (the distribution *assuming* the null hypothesis) then we reject  $H_0$ .

In the case where there is no special value to test we may still want to estimate  $\theta$ . This is the reverse of significance testing; rather than seeing if we should reject a specific value of  $\theta$  because it doesn't fit the data we want to find the range of values of  $\theta$  that do, in some sense, fit the data. This gives us the following definitions.

**Definition.** Given a value  $x$  of the test statistic, the  $(1 - \alpha)$  confidence interval contains all values  $\theta_0$  which are not rejected (at significance level  $\alpha$ ) when they are the null hypothesis.

**Definition.** A type 1 CI error occurs when the confidence interval does not contain the true value of  $\theta$ .

For a  $(1 - \alpha)$  confidence interval the type 1 CI error rate is  $\alpha$ .

**Example 1.** Here is an example relating confidence intervals and hypothesis tests. Suppose data  $x$  is drawn from a binomial(12,  $\theta$ ) distribution with  $\theta$  unknown. Let  $\alpha = .1$  and create the  $(1 - \alpha) = 90\%$  confidence interval for each possible value of  $x$ .

Our strategy is to look at one possible value of  $\theta$  at a time and choose rejection regions for a significance test with  $\alpha = .1$ . Once this is done, we will know, for each value of  $x$ , which values of  $\theta$  are not rejected, i.e. the confidence interval associated with  $x$ .

To start we set up a likelihood table for binomial(12,  $\theta$ ) in Table 1. Each row shows the probabilities  $p(x|\theta)$  for one value of  $\theta$ . To keep the size manageable we only show  $\theta$  in increments of 0.1.

$\theta/x$	0	1	2	3	4	5	6	7	8	9	10	11	12
1.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00
0.9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.09	0.23	0.38	0.28
0.8	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.05	0.13	0.24	0.28	0.21	0.07
0.7	0.00	0.00	0.00	0.00	0.01	0.03	0.08	0.16	0.23	0.24	0.17	0.07	0.01
0.6	0.00	0.00	0.00	0.01	0.04	0.10	0.18	0.23	0.21	0.14	0.06	0.02	0.00
0.5	0.00	0.00	0.02	0.05	0.12	0.19	0.23	0.19	0.12	0.05	0.02	0.00	0.00
0.4	0.00	0.02	0.06	0.14	0.21	0.23	0.18	0.10	0.04	0.01	0.00	0.00	0.00
0.3	0.01	0.07	0.17	0.24	0.23	0.16	0.08	0.03	0.01	0.00	0.00	0.00	0.00
0.2	0.07	0.21	0.28	0.24	0.13	0.05	0.02	0.00	0.00	0.00	0.00	0.00	0.00
0.1	0.28	0.38	0.23	0.09	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.0	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

**Table 1.** Likelihood table for Binomial(12,  $\theta$ )

Tables 2-4 below show the rejection region (in orange) and non-rejection region (in blue) for the various values of  $\theta$ . To emphasize the row-by-row nature of the process the Table 2 just shows these regions for  $\theta = 1.0$ , then Table 3 adds in regions for  $\theta = 0.9$  and Table 4 shows them for all the values of  $\theta$ .

Immediately following the tables we give a detailed explanation of how the rejection/non-rejection regions were chosen.

$\theta/x$	0	1	2	3	4	5	6	7	8	9	10	11	12	significance
1.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.000
0.9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.09	0.23	0.38	0.28	
0.8	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.05	0.13	0.24	0.28	0.21	0.07	
0.7	0.00	0.00	0.00	0.00	0.01	0.03	0.08	0.16	0.23	0.24	0.17	0.07	0.01	
0.6	0.00	0.00	0.00	0.01	0.04	0.10	0.18	0.23	0.21	0.14	0.06	0.02	0.00	
0.5	0.00	0.00	0.02	0.05	0.12	0.19	0.23	0.19	0.12	0.05	0.02	0.00	0.00	
0.4	0.00	0.02	0.06	0.14	0.21	0.23	0.18	0.10	0.04	0.01	0.00	0.00	0.00	
0.3	0.01	0.07	0.17	0.24	0.23	0.16	0.08	0.03	0.01	0.00	0.00	0.00	0.00	
0.2	0.07	0.21	0.28	0.24	0.13	0.05	0.02	0.00	0.00	0.00	0.00	0.00	0.00	
0.1	0.28	0.38	0.23	0.09	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
0.0	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	

**Table 2.** Likelihood table for binomial(12,  $\theta$ ) with rejection/non-rejection regions for  $\theta = 1.0$

$\theta/x$	0	1	2	3	4	5	6	7	8	9	10	11	12	significance
1.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.000
0.9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.09	0.23	0.38	0.28	0.026
0.8	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.05	0.13	0.24	0.28	0.21	0.07	
0.7	0.00	0.00	0.00	0.00	0.01	0.03	0.08	0.16	0.23	0.24	0.17	0.07	0.01	
0.6	0.00	0.00	0.00	0.01	0.04	0.10	0.18	0.23	0.21	0.14	0.06	0.02	0.00	
0.5	0.00	0.00	0.02	0.05	0.12	0.19	0.23	0.19	0.12	0.05	0.02	0.00	0.00	
0.4	0.00	0.02	0.06	0.14	0.21	0.23	0.18	0.10	0.04	0.01	0.00	0.00	0.00	
0.3	0.01	0.07	0.17	0.24	0.23	0.16	0.08	0.03	0.01	0.00	0.00	0.00	0.00	
0.2	0.07	0.21	0.28	0.24	0.13	0.05	0.02	0.00	0.00	0.00	0.00	0.00	0.00	
0.1	0.28	0.38	0.23	0.09	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
0.0	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	

**Table 3.** Likelihood table with rejection/non-rejection regions shown for  $\theta = 1.0$  and 0.9

$\theta/x$	0	1	2	3	4	5	6	7	8	9	10	11	12	significance
1.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.000
0.9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.09	0.23	0.38	0.28	0.026
0.8	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.05	<b>0.13</b>	0.24	0.28	0.21	0.07	0.073
0.7	0.00	0.00	0.00	0.00	0.01	0.03	0.08	0.16	<b>0.23</b>	0.24	0.17	0.07	0.01	0.052
0.6	0.00	0.00	0.00	0.01	0.04	0.10	0.18	0.23	<b>0.21</b>	0.14	0.06	0.02	0.00	0.077
0.5	0.00	0.00	0.02	0.05	0.12	0.19	0.23	0.19	<b>0.12</b>	0.05	0.02	0.00	0.00	0.092
0.4	0.00	0.02	0.06	0.14	0.21	0.23	0.18	0.10	0.04	0.01	0.00	0.00	0.00	0.077
0.3	0.01	0.07	0.17	0.24	0.23	0.16	0.08	0.03	0.01	0.00	0.00	0.00	0.00	0.052
0.2	0.07	0.21	0.28	0.24	0.13	0.05	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.073
0.1	0.28	0.38	0.23	0.09	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.026
0.0	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.000

**Table 4.** Likelihood table with rejection/non-rejection regions for  $\theta = 0.0$  to 1.0

### Choosing the rejection and non-rejection regions in the tables

The first problem we confront is how exactly to choose the rejection region. We used two rules:

1. The total probability of the rejection region, i.e. the significance, should be less than or equal to 0.1. (Since we have a discrete distribution it is impossible to make the significance exactly 0.1.)
2. We build the rejection region by choosing values of  $x$  one at a time, always picking the unused value with the smallest probability. We stop when the next value would make the significance more than 0.1.

There are other ways to choose the rejection region which would result in slight differences. Our method is one reasonable way.

Table 2 shows the rejection (orange) and non-rejection (blue) regions for  $\theta = 1.0$ . This is a special case because most of the probabilities in this row are 0.0. We'll move right on to the next table and step through the process for that.

In Table 3, let's walk through the steps used to find these regions for  $\theta = 0.9$ .

- The smallest probability is when  $x = 0$ , so  $x = 0$  is in the rejection region.
- The next smallest is when  $x = 1$ , so  $x = 1$  is in the rejection region.
- We continue with  $x = 2, \dots, 8$ . At this point the total probability in the rejection region is 0.026.
- The next smallest probability is when  $x = 9$ . Adding this probability (0.09) to 0.026 would put the total probability over 0.1. So we leave  $x = 9$  out of the rejection region and stop the process.

Note three things for the  $\theta = 0.9$  row:

1. None of the probabilities in this row are truly zero, though some are small enough that they equal 0 to 2 decimal places.
2. We show the significance for this value of  $\theta$  in the right hand margin. More precisely, we show the significance level of the NHST with null hypothesis  $\theta = 0.9$  and the given rejection region.
3. The rejection region consists of values of  $x$ . When we say the rejection region is shown in orange we really mean the rejection region contains the values of  $x$  corresponding to the probabilities highlighted in orange.

**Think:** Look back at the  $\theta = 1.0$  row and make sure you understand why the rejection region is  $x = 0, \dots, 11$  and the significance is 0.000.

**Example 2.** Using Table 4 determine the 0.90 confidence interval when  $x = 8$ .

**answer:** The 90% confidence interval consists of all those  $\theta$  that would not be rejected by an  $\alpha = 0.1$  hypothesis test when  $x = 8$ . Looking at the table, the blue (non-rejected) entries in the column  $x = 8$  correspond to  $0.5 \leq \theta \leq 0.8$ : the confidence interval is  $[0.5, 0.8]$ .

**Remark:** The point of this example is to show how confidence intervals and hypothesis tests are related. Since Table 4 has only finitely many values of  $\theta$ , our answer is close but not exact. Using a computer we could look at many more values of  $\theta$ . For this problem we used R to find that, correct to 2 decimal places, the confidence interval is  $[0.42, 0.85]$ .

**Example 3.** Explain why the expected type one CI error rate will be at most 0.092, provided that the true value of  $\theta$  is in the table.

**answer:** The short answer is that this is the maximum significance for any  $\theta$  in Table 4. Expanding on that slightly: we make a type one CI error if the confidence interval does not contain the true value of  $\theta$ , call it  $\theta_{\text{true}}$ . This happens exactly when the data  $x$  is in the rejection region for  $\theta_{\text{true}}$ . The probability of this happening is the significance for  $\theta_{\text{true}}$  and this is at most 0.092.

**Remark:** The point of this example is to show how confidence level, type one CI error rate and significance for each hypothesis are related. As in the previous example, we can use R to compute the significance for many more values of  $\theta$ . When we do this we find that the maximum significance for any  $\theta$  is 0.1 occurring when  $\theta \approx 0.0452$ .

#### Summary notes:

1. We start with a test statistic  $x$ . The confidence interval is random because it depends on  $x$ .
2. For each hypothesized value of  $\theta$  we make a significance test with significance level  $\alpha$  by choosing rejection regions.
3. For a specific value of  $x$  the associated confidence interval for  $\theta$  consists of all  $\theta$  that aren't rejected for that value, i.e. all  $\theta$  that have  $x$  in their non-rejection regions.
4. Because the distribution is discrete we can't always achieve the exact significance level, so our confidence interval is really an 'at least 90% confidence interval'.

**Example 4.** Open the applet <http://ocw.mit.edu/ans7870/18/18.05/s14/applets/confidence-jmo.html>. We want you to play with the applet to understand the random nature of confidence intervals and the meaning of confidence as (1 - type I CI error rate).

- (a) Read the help. It is short and will help orient you in the applet. Play with different settings of the parameters to see how they affect the size of the confidence intervals.
- (b) Set the number of trials to  $N = 1$ . Click the 'Run N trials' button repeatedly and see that each time data is generated the confidence intervals jump around.
- (c) Now set the confidence level to  $c = .5$ . As you click the 'Run N trials' button you should see that about 50% of the confidence intervals include the true value of  $\mu$ . The 'Z correct' and 't correct' values should change accordingly.
- (d) Now set the number of trials to  $N = 100$ . With  $c = .8$ . The 'Run N trials' button will now run 100 trials at a time. Only the last confidence interval will be shown in the graph, but the trials all run and the 'percent correct' statistics will be updated based on all 100 trials.

Click the run trials button repeatedly. Watch the correct rates start to converge to the confidence level. To converge even faster, set  $N = 1000$ .

## 5 Formal view of confidence intervals

Recall: An interval statistic is an interval  $I_x$  computed from data  $x$ . An interval is determined by its lower and upper bounds, and these are random because  $x$  is random.

We suppose that  $x$  is drawn from a distribution with pdf  $f(x|\theta)$  where the parameter  $\theta$  is unknown.

**Definition:** A  $(1 - \alpha)$  confidence interval for  $\theta$  is an interval statistic  $I_x$  such that

$$P(I_x \text{ contains } \theta_0 \mid \theta = \theta_0) = 1 - \alpha$$

for all possible values of  $\theta_0$ .

We wish this was simpler, but a definition is a definition and this definition is one way to weigh the evidence provided by the data  $x$ . Let's unpack it a bit.

The confidence level of an interval statistic is a probability concerning a random interval and a hypothesized value  $\theta_0$  for the unknown parameter. Precisely, it is the probability that the random interval (computed from random data) contains the value  $\theta_0$ , *given that the model parameter truly is  $\theta_0$* . Since the true value of  $\theta$  is unknown, the frequentist statistician defines confidence intervals so that this .95 probability is valid *no matter which hypothesized value of the parameter is actually true*.

**Note:** Equality in this equation is often relaxed to  $\geq$  or  $\approx$ . We have:

1.  $=$  :  $z$ ,  $t$ ,  $\chi^2$  confidence intervals
2.  $\geq$  : rule-of-thumb and exact binomial (polling)
3.  $\approx$  : large sample confidence interval

## 6 Comparison with Bayesian probability intervals

Confidence intervals are a frequentist notion, and as we've repeated many times, frequentists don't assign probabilities to hypotheses, e.g. the value of an unknown parameter. Rather they compute likelihoods; that is, probabilities about data or associated statistics given a hypothesis (note the condition  $\theta = \theta_0$  in the formal view of confidence intervals). Note that the construction of confidence intervals proceeds entirely from the full likelihood table.

In contrast Bayesian posterior probability intervals are truly the probability that the value of the unknown parameter lies in the reported range. We add the usual caveat that this depends on the specific choice of a (possibly subjective) Bayesian prior.

This distinction between the two is subtle because Bayesian posterior probability intervals and frequentist confidence intervals share the following properties:

1. They start from a model  $f(x|\theta)$  for observed data  $x$  with unknown parameter  $\theta$ .
2. Given data  $x$ , they give an interval  $I(x)$  specifying a range of values for  $\theta$ .
3. They come with a number (say .95) that is the probability of something.

In practice, many people misinterpret confidence intervals as Bayesian probability intervals, forgetting that frequentists *never* place probabilities on hypotheses (this is analogous to mistaking the  $p$ -value in NHST for the probability that  $H_0$  is false). The harm of this misinterpretation is somewhat mitigated by that fact that, given enough data and a reasonable prior, Bayesian and frequentist intervals often work out to be quite similar.

For an amusing example illustrating how they can be quite different, see the first answer here (involving chocolate chip cookies!):

<http://stats.stackexchange.com/questions/2272/whats-the-difference-between-a-confidence-interval-and-a-credible-interval>

This example uses the formal definitions and is really about confidence sets instead of confidence intervals.

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