Exam 1 Practice Questions II –solutions, 18.05, Spring 2014

Note: This is a set of practice problems for exam 1. The actual exam will be much shorter.

1. We build a full-house in stages and count the number of ways to make each stage:

Stage 1. Choose the rank of the pair: $\binom{13}{1}$.

Stage 2. Choose the pair from that rank, i.e. pick 2 of 4 cards: $\binom{4}{2}$.

Stage 3. Choose the rank of the triple (from the remaining 12 ranks): $\binom{12}{1}$.

Stage 4. Choose the triple from that rank: $\binom{4}{3}$.

Number of ways to get a full-house: $\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{3}$

Number of ways to pick any 5 cards out of 52: $\binom{52}{5}$

Probability of a full house: $\dfrac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{14}{3}}{\binom{52}{5}} \approx 0.00144$

2. **(a)** There are $\binom{20}{3}$ ways to choose the 3 people to set the table, then $\binom{17}{2}$ ways to choose the 2 people to boil water, and $\binom{15}{6}$ ways to choose the people to make scones. So the total number of ways to choose people for these tasks is

$$\boxed{\binom{20}{3}\binom{17}{2}\binom{15}{6} = \frac{20!}{3!\,17!}\cdot\frac{17!}{2!\,15!}\cdot\frac{15!}{6!\,9!} = \frac{20!}{3!\,2!\,6!\,9!} = 775975200.}$$

(b) The number of ways to choose 10 of the 20 people is $\binom{20}{10}$ The number of ways to choose 10 people from the 14 Republicans is $\binom{14}{10}$. So the probability that you only choose 10 Republicans is

$$\frac{\binom{14}{10}}{\binom{20}{10}} = \frac{\frac{14!}{10!\,4!}}{\frac{20!}{10!\,10!}} \approx 0.00542$$

Alternatively, you could choose the 10 people in sequence and say that there is a 14/20 probability that the first person is a Republican, then a 13/19 probability that the second one is, a 12/18 probability that third one is, etc. This gives a probability of

$$\frac{14}{20}\cdot\frac{13}{19}\cdot\frac{12}{18}\cdot\frac{11}{17}\cdot\frac{10}{16}\cdot\frac{9}{15}\cdot\frac{8}{14}\cdot\frac{7}{13}\cdot\frac{6}{12}\cdot\frac{5}{11}.$$

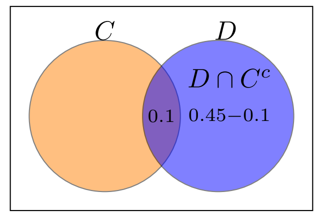
(You can check that this is the same as the other answer given above.)

(c) You can choose 1 Democrat in $\binom{6}{1} = 6$ ways, and you can choose 9 Republicans in $\binom{14}{9}$ ways, so the probability equals

$$\frac{6 \cdot \binom{14}{9}}{\binom{20}{10}} = \frac{6 \cdot \frac{14!}{9!\,5!}}{\frac{20!}{10!\,10!}} = \frac{6 \cdot 14!\,10!\,10!}{9!\,5!\,20!}.$$

3. D is the disjoint union of $D \cap C$ and $D \cap C^c$.

So, $P(D \cap C) + P(D \cap C^c) = P(D)$
$\Rightarrow\ P(D \cap C^c) = P(D) - P(D \cap C) = 0.45 - 0.1 = \boxed{0.35.}$
(We never use $P(C) = 0.25$.)



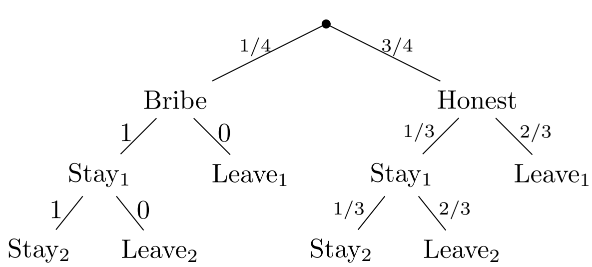
4. Let H_i be the event that the i^{th} hand has one king. We have the conditional probabilities

$$P(H_1) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}}; \quad P(H_2|H_1) = \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}}; \quad P(H_3|H_1 \cap H_2) = \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}}$$

$P(H_4|H_1 \cap H_2 \cap H_3) = 1$

$$\begin{aligned} P(H_1 \cap H_2 \cap H_3 \cap H_4) &= P(H_4|H_1 \cap H_2 \cap H_3)\, P(H_3|H_1 \cap H_2)\, P(H_2|H_1)\, P(H_1) \\ &= \frac{\binom{2}{1}\binom{24}{12}\binom{3}{1}\binom{36}{12}\binom{4}{1}\binom{48}{12}}{\binom{26}{13}\binom{39}{13}\binom{52}{13}}. \end{aligned}$$

5. The following tree shows the setting. Stay$_1$ means the contestant was allowed to stay during the first episode and stay$_2$ means the they were allowed to stay during the second.



Let's name the relevant events:
B = the contestant is bribing the judges

H = the contestant is honest (not bribing the judges)
S_1 = the contestant was allowed to stay during the first episode
S_2 = the contestant was allowed to stay during the second episode
L_1 = the contestant was asked to leave during the first episode
L_2 = the contestant was asked to leave during the second episode

(a) We first compute $P(S_1)$ using the law of total probability.

$$P(S_1) = P(S_1|B)P(B) + P(S_1|H)P(H) = 1 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{2}.$$

We therefore have (by Bayes' rule) $P(B|S_1) = P(S_1|B)\dfrac{P(B)}{P(S_1)} = 1 \cdot \dfrac{1/4}{1/2} = \dfrac{1}{2}.$

(b) Using the tree we have the total probability of S_2 is

$$P(S_2) = \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{3}$$

(c) We want to compute $P(L_2|S_1) = \dfrac{P(L_2 \cap S_1)}{P(S_1)}.$

From the calculation we did in part (a), $P(S_1) = 1/2$. For the numerator, we have (see the tree)

$$P(L_2 \cap S_1) = P(L_2 \cap S_1|B)P(B) + P(L_2 \cap S_1|H)P(H) = 0 \cdot \frac{1}{3} + \frac{2}{9} \cdot \frac{3}{4} = \frac{1}{6}$$

Therefore $\boxed{P(L_2|S_1) = \dfrac{1/6}{1/2} = \dfrac{1}{3}.}$

6. You should write this out in a tree! (For example, see the solution to the next problem.)

We compute all the pieces needed to apply Bayes' rule. We're given
$P(T|D) = 0.9 \Rightarrow P(T^c|D) = 0.1, \quad P(T|D^c) = 0.01 \Rightarrow P(T^c|D^c) = 0.99.$
$P(D) = 0.0005 \Rightarrow P(D^c) = 1 - P(D) = 0.9995.$

We use the law of total probability to compute $P(T)$:

$$P(T) = P(T|D)\,P(D) + P(T|D^c)\,P(D^c) = 0.9 \cdot 0.0005 + 0.01 \cdot 0.9995 = 0.010445$$

Now we can use Bayes' rule to answer the questions:

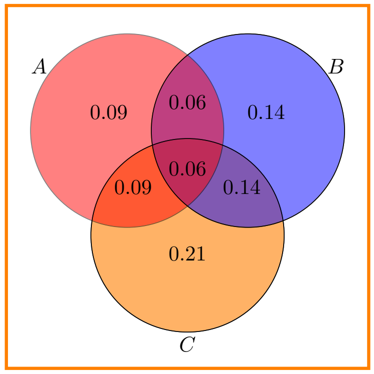
$$P(D|T) = \frac{P(T|D)\,P(D)}{P(T)} = \frac{0.9 \times 0.0005}{0.010445} = 0.043$$

$$P(D|T^c) = \frac{P(T^c|D)\,P(D)}{P(T^c)} = \frac{0.1 \times 0.0005}{0.989555} = 5.0 \times 10^{-5}$$

7. By the mutual independence we have

$$P(A \cap B \cap C) = P(A)P(B)P(C) = 0.06 \qquad P(A \cap B) = P(A)P(B) = 0.12$$
$$P(A \cap C) = P(A)P(C) = 0.15 \qquad P(B \cap C) = P(B)P(C) = 0.2$$

We show this in the following Venn diagram



Note that, for instance, $P(A \cap B)$ is split into two pieces. One of the pieces is $P(A \cap B \cap C)$ which we know and the other we compute as $P(A \cap B) - P(A \cap B \cap C) = 0.12 - 0.06 = 0.06$. The other intersections are similar.

We can read off the asked for probabilities from the diagram.
(i) $P(A \cap B \cap C^c) = 0.06$
(ii) $P(A \cap B^c \cap C) = 0.09$
(iii) $P(A^c \cap B \cap C) = 0.14$.

8. Use $\text{Var}(X) = E(X^2) - E(X)^2 \Rightarrow 2 = E(X^2) - 25 \Rightarrow \boxed{E(X^2) = 27.}$

9. **(a)** It is easy to see that (e.g. look at the probability tree) $P(2^k) = \dfrac{1}{2^{k+1}}$.

(b) $E(X) = \displaystyle\sum_{k=0}^{\infty} 2^k \frac{1}{2^{k+1}} = \sum \frac{1}{2} = \infty$. Technically, $E(X)$ is undefined in this case.

(c) Technically, $E(X)$ is undefined in this case. But the value of ∞ tells us what is wrong with the scheme. Since the average last bet is infinite, I need to have an infinite amount of money in reserve.

This problem and solution is often referred to as the **St. Petersburg paradox**

10. **(a)** We have cdf of X,

$$F_X(x) = \int_0^x \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}.$$

Now for $y \geq 0$, we have

(b)

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = 1 - e^{-\lambda\sqrt{y}}.$$

Differentiating $F_Y(y)$ with respect to y, we have

$$f_Y(y) = \frac{\lambda}{2} y^{-\frac{1}{2}} \mathrm{e}^{-\lambda\sqrt{y}}.$$

11. **(a)** Note: $Y = 1$ when $X = 1$ or $X = -1$, so

$$P(Y = 1) = P(X = 1) + P(X = -1).$$

Values y of Y	0	1	4
pmf $p_Y(y)$	3/15	6/15	6/15

(b) and **(c)** To distinguish the distribution functions we'll write F_x and F_Y.

Using the tables in part (a) and the definition $F_X(a) = P(X \le a)$ etc. we get

a	-1.5	3/4	7/8	1	1.5	5
$F_X(a)$	1/15	6/15	6/15	10/15	10/15	1
$F_Y(a)$	0	3/15	3/15	9/15	9/15	1

12.
(i) yes, discrete, (ii) no, (iii) no, (iv) no, (v) yes, continuous
(vi) no (vii) yes, continuous, (viii) yes, continuous.

13. Normal Distribution: **(a)** We have

$$F_X(x) = P(X \le x) = P(3Z + 1 \le x) = P(Z \le \frac{x-1}{3}) = \Phi\left(\frac{x-1}{3}\right).$$

(b) Differentiating with respect to x, we have

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{dx}} F_X(x) = \frac{1}{3}\phi\left(\frac{x-1}{3}\right).$$

Since $\phi(x) = (2\pi)^{-\frac{1}{2}} \mathrm{e}^{-\frac{x^2}{2}}$, we conclude

$$f_X(x) = \frac{1}{3\sqrt{2\pi}} \mathrm{e}^{-\frac{(x-1)^2}{2\cdot 3^2}},$$

which is the probability density function of the $N(1, 9)$ distribution. **Note:** The arguments in (a) and (b) give a proof that $3Z + 1$ is a normal random variable with mean 1 and variance 9. See Problem Set 3, Question 5.

(c) We have

$$P(-1 \le X \le 1) = P\left(-\frac{2}{3} \le Z \le 0\right) = \Phi(0) - \Phi\left(-\frac{2}{3}\right) \approx 0.2475$$

(d) Since $E(X) = 1$, $\text{Var}(X) = 9$, we want $P(-2 \le X \le 4)$. We have

$$P(-2 \le X \le 4) = P(-3 \le 3Z \le 3) = P(-1 \le Z \le 1) \approx 0.68.$$

14. The density for this distribution is $f(x) = \lambda\, e^{-\lambda x}$. We know (or can compute) that the distribution function is $F(a) = 1 - e^{-\lambda a}$. The median is the value of a such that $F(a) = .5$. Thus, $1 - e^{-\lambda a} = 0.5 \;\Rightarrow\; 0.5 = e^{-\lambda a} \;\Rightarrow\; \log(0.5) = -\lambda a \;\Rightarrow$ $\boxed{a = \log(2)/\lambda.}$

15. **(a)** First note by linearity of expectation we have $E(X+s) = E(X) + s$, thus $X + s - E(X+s) = X - E(X)$.
Likewise $Y + u - E(Y+u) = Y - E(Y)$.
Now using the definition of covariance we get

$$\begin{aligned}\text{Cov}(X+s, Y+u) &= E((X+s-E(X+s)) \cdot (Y+u-E(Y+u))) \\ &= E((X-E(X)) \cdot (Y-E(Y))) \\ &= \text{Cov}(X, Y).\end{aligned}$$

(b) This is very similar to part (a).

We know $E(rX) = rE(X)$, so $rX - E(rX) = r(X - E(X))$. Likewise $tY - E(tY) = s(Y - E(Y))$. Once again using the definition of covariance we get

$$\begin{aligned}\text{Cov}(rX, tY) &= E((rX - E(rX))(tY - E(tY))) \\ &= E(rt(X - E(X))(Y - E(Y))) \\ &\qquad \text{(Now we use linearity of expectation to pull out the factor of } rt) \\ &= rtE((X - E(X)(Y - E(Y)))) \\ &= rt\text{Cov}(X, Y)\end{aligned}$$

(c) This is more of the same. We give the argument with far fewer algebraic details

$$\begin{aligned}\text{Cov}(rX+s, tY+u) &= \text{Cov}(rX, tY) \text{ (by part (a))} \\ &= rt\text{Cov}(X, Y) \text{ (by part (b))}\end{aligned}$$

16. (**Another Arithmetic Puzzle**)
(a) $U = X + Y$ takes values 0, 1, 2 and $V = X - Y$ takes values -1, 0, 1.

First we make two tables: the joint probability table for X and Y and a table given the values (S, T) corresponding to values of (X, Y), e.g. $(X, Y) = (1, 1)$ corresponds to $(S, T) = (2, 0)$.

$X \backslash Y$	0	1
0	1/4	1/4
1	1/4	1/4

Joint probabilities of X and Y

$X \backslash Y$	0	1
0	0,0	0,-1
1	1,1	2,0

Values of (S, T) corresponding to X and Y

We can use the two tables above to write the joint probability table for S and T. The marginal probabilities are given in the table.

$S \backslash T$	-1	0	1	
0	1/4	1/4	0	1/2
1	0	0	1/4	1/4
2	0	0	1/4	1/4
	1/4	1/4	1/2	1

Joint and marginal probabilities of S and T

(b) No probabilities in the table are the product of the corresponding marginal probabilities. (This is easiest to see for the 0 entries.) So, S and T are not independent

17. **(a)** X and Y are independent, so the table is computed from the product of the known marginal probabilities. Since they are independent, $\text{Cov}(X, Y) = 0$.

$Y \backslash X$	0	1	P_Y
0	1/8	1/8	1/4
1	1/4	1/4	1/2
2	1/8	1/8	1/4
P_X	1/2	1/2	1

(b) The sample space is $\Omega = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$.

$P(X = 0, F = 0) = P(\{TTH, TTT\}) = 1/4$.

$P(X = 0, F = 1) = P(\{THH, THT\}) = 1/4$.

$P(X = 0, F = 2) = 0$.

$P(X = 1, F = 0) = 0$.

$P(X = 1, F = 1) = P(\{HTH, HTT\}) = 1/4$.

$P(X = 1, F = 2) = P(\{HHH, HHT\}) = 1/4$.

$F \backslash X$	0	1	P_F
0	1/4	0	1/4
1	1/4	1/4	1/2
2	0	1/4	1/4
P_X	1/2	1/2	1

$\text{Cov}(X, F) = E(XF) - E(X)E(F)$.

$E(X) = 1/2, \quad E(F) = 1,\ E(XF) = \sum x_i y_j p(x_i, y_j) = 3/4$.

$\Rightarrow \ \text{Cov}(X, F) = 3/4 - 1/2 = \boxed{1/4.}$

18. (**More Central Limit Theorem**)

Let X_j be the IQ of a randomly selected person. We are given $E(X_j) = 100$ and $\sigma_{X_j} = 15$.

Let $\overline{X}$ be the average of the IQ's of 100 randomly selected people. Then we know

$$E(\overline{X}) = 100 \quad \text{and} \quad \sigma_{\overline{X}} = 15/\sqrt{100} = 1.5.$$

The problem asks for $P(\overline{X} > 115)$. Standardizing we get $P(\overline{X} > 115) \approx P(Z > 10)$.

$\boxed{\text{This is effectively 0.}}$